

U(1) Holonomy Algebra on Null Initial Surfaces

Wolfgang WIELAND with Carolina Sole PANELLA

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Wydział Fizyki UW and DBP NCBJ, Warsaw, Poland

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Friedrich-Alexander-Universität
Erlangen-Nürnberg



- **Motivation:** Charge network representation of the quantum Maxwell field as a toy model of loop quantum gravity
- Covariant phase space of **local subsystems on null initial surfaces**
- **Holonomy algebra** and its quantisation

Motivation: Charge Network Representation of the
Quantum Maxwell Field

In what follows, we work on a fixed background (\mathcal{M}, g_{ab}) . We consider the algebra of the Wilson loop observables for a $U(1)$ gauge field.

Action:

$$S[{}^4A] = -\frac{1}{4e^2} \int_{\mathcal{M}} d^4v_g {}^4F_{ab} {}^4F^{ab},$$

where ${}^4F_{ab} = 2\nabla_{[a} {}^4A_{b]}$ is the field strength.

Variation and Hamilton-Jacobi equation:

$$\delta S[{}^4A] = \frac{1}{e^2} \int_{\mathcal{M}} (d \star {}^4F) \wedge \delta {}^4A - \frac{1}{e^2} \int_{\partial\mathcal{M}} \star {}^4F \wedge \delta {}^4A.$$

Pre-symplectic current and potential:

$$\begin{aligned}\theta &= -e^{-2} \star {}^4F \wedge \mathbb{d}{}^4A, \\ \Theta_{\Sigma} &= \int_{\Sigma} \theta = -e^{-2} \int_{\Sigma} \tilde{E}^a \mathbb{d}A_a.\end{aligned}$$

On space-like initial hypersurface, we obtain the Poisson brackets

$$\{\tilde{E}^a(x), A_b(y)\}_\Sigma = -e^2 h^a_b \tilde{\delta}_\Sigma(x, y),$$

where $A_a = \varphi_\Sigma^* A_a$ (vector potential) and $\tilde{E}^a = d^3 v_h {}^4 F^a_b n^b|_\Sigma$ (electric field) and $h_{ab} = g_{ab} + n_a n_b$ (projector onto Σ).

Gauss law: physical phase space obtained by imposing the constraint

$$D_a \tilde{E}^a = 0.$$

Electric and (compactified) magnetic fluxes:

$$\mathcal{E}_\Delta[E] = \int_\Delta E,$$
$$h_\gamma[A] = e^{-i \int_\gamma A}.$$

Electric and magnetic fluxes form a closed algebra.

$$\{\mathcal{E}_\Delta, h_\gamma\}_\Sigma = -ie^2 \varepsilon(\Delta, \gamma) h_\gamma,$$

If we restrict ourselves to a fixed graph $\Gamma = (\gamma, \gamma', \dots)$ this is simply the phase space $T^*U(1)^{|\Gamma|}$.

Charge network states:

$$\begin{aligned}\hat{h}_{\gamma_i} |n_1, \dots, n_N\rangle &= |n_1, \dots, n_i + 1, \dots, n_N\rangle, \\ \hat{\mathcal{E}}_\Delta |n_1, \dots, n_N\rangle &= -\hbar e^2 \sum_i n_i \varepsilon(\Delta, \gamma_i) |n_1, \dots, n_N\rangle.\end{aligned}$$

Gauss constraint: Physical states Ψ_{phys} satisfy the Gauss constraint:

$$\forall f : \partial f = \emptyset : \hat{\mathcal{E}}_f \Psi_{phys} = 0.$$

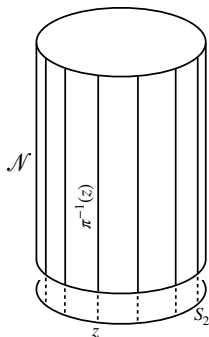
The construction is cylindrical consistent, i.e. stable under graph refinements. The resulting Hilbert space can be understood as an L^2 -space over the space of generalised $U(1)$ connections [Ashtekar, Lewandowski,...].

What happens if we work on a null surface instead?

Covariant Phase Space of Local Subsystems on Null Initial Surfaces

Spacetime region bounded by null surface:

- Compact spacetime region \mathcal{M} .
- Bounded by spacelike disks M_0, M_1 and null surface \mathcal{N} .
- Null surface boundary \mathcal{N} embedded into abstract bundle (ruled surface)
 $P(\pi, \mathcal{E}) \simeq \mathbb{R} \times \mathcal{E}$.
- Null generators $\pi^{-1}(z)$.



Introduce double null foliation in vicinity of \mathcal{N} .

Adapted Newman–Penrose null tetrad

$$g_{ab} = -2^4 \ell_{(a} {}^4 k_{b)} + 2^4 m_{(a} {}^4 \bar{m}_{b)},$$

such that

$$\varphi_{\mathcal{N}}^*({}^4 \ell_a, {}^4 k_a, {}^4 m_a, {}^4 \bar{m}_a) = (0, k_a, m_a, \bar{m}_a).$$

Without loss of generality, ${}^4 k_a = -\nabla_a u$, ${}^4 \ell_{[a} \nabla_b {}^4 \ell_{c]} = 0$.

Anholonomic basis, exterior derivatives define $U(1)$ **spin connection**, **shear** and **expansion**, **inaffinity**

$$\begin{aligned} d^4 m &= -\frac{1}{2} \vartheta_{(k)} {}^4 \ell \wedge {}^4 m - \frac{1}{2} \vartheta_{(\ell)} {}^4 k \wedge {}^4 m - i {}^4 \Gamma \wedge {}^4 m + \\ &\quad - \sigma_{(\ell)} {}^4 k \wedge {}^4 \bar{m} - \sigma_{(k)} {}^4 \ell \wedge {}^4 \bar{m} + \tau {}^4 k \wedge {}^4 \ell, \\ d^4 \ell &= -\psi \wedge {}^4 \ell, \quad \kappa_{(\ell)} = -{}^4 \ell^a \psi_a. \end{aligned}$$

Vector potential in neighbourhood of \mathcal{N}

$${}^4A_a = V^4\ell_a + \bar{\alpha}^4m_a + \alpha^4\bar{m}_a + \nabla_a\lambda.$$

Magnetic fields on \mathcal{N}

$$\begin{aligned} F = dA &= \varphi_{\mathcal{N}}^* {}^4F = \\ &= + (L_{\bar{m}}\alpha + i\bar{\gamma}\alpha - \text{cc.}) m \wedge \bar{m} \\ &\quad - \left[(L_{\ell}\alpha + \left(\frac{1}{2}\vartheta_{(\ell)} - i\varphi_{(\ell)}\right) \alpha + \sigma_{(\ell)}\bar{\alpha}) k \wedge \bar{m} + \text{cc.} \right], \end{aligned}$$

Electric fields on \mathcal{N}

$$\begin{aligned} \star F := \varphi_{\mathcal{N}}^*[\star {}^4F] &= -i(L_{\ell}V + \kappa_{(\ell)}V - (\bar{\tau}\alpha + \tau\bar{\alpha})) m \wedge \bar{m} + \\ &\quad - i \left[(L_{\ell}\alpha + \left(\frac{1}{2}\vartheta_{(\ell)} - i\varphi_{(\ell)}\right) \alpha + \sigma_{(\ell)}\bar{\alpha}) k \wedge \bar{m} - \text{cc.} \right]. \end{aligned}$$

NB: electric and magnetic fields no longer independent. Immediate consequence of self-duality of the two-form $\star({}^4k \wedge {}^4m) = -i{}^4k \wedge {}^4m$.

Pre-symplectic potential

$$\Theta_{\mathcal{N}} = -e^{-2} \int_{\mathcal{N}} \star^4 F \wedge \mathbb{d}A.$$

Vector potential on \mathcal{N}

$$\varphi_{\mathcal{N}}^* A_a = A_a = \bar{\alpha} m_a + \alpha \bar{m}_a + \partial_a \lambda.$$

Bulk-corner split

$$\Theta_{\mathcal{N}} = \Theta_{\mathcal{N}}^{rad} + \Theta_{\mathcal{E}_+}^{edge} - \Theta_{\mathcal{E}_-}^{edge}.$$

Bulk and boundary symplectic structure

$$\Theta_{\mathcal{N}}^{rad} = -\frac{1}{e^2} \int_{\mathcal{N}} \left[i (\ell[\alpha] + (\frac{1}{2}\vartheta_{(\ell)} - i\varphi_{(\ell)}) \alpha + \sigma_{(\ell)} \bar{\alpha}) \, \mathbf{d}\bar{\alpha} \, k \wedge m \wedge \bar{m} + \text{cc.} \right],$$

$$\Theta_{\mathcal{E}}^{edge} = \frac{i}{e^2} \oint_{\mathcal{E}} \left[\ell[V] + \kappa_{(\ell)} V - (\bar{\tau}\alpha + \tau\bar{\alpha}) \right] m \wedge \bar{m} \, \mathbf{d}\lambda.$$

Shear cancels from symplectic two-form

$$\begin{aligned} \Omega_{\mathcal{N}}^{rad} &= \mathbf{d}\Theta_{\mathcal{N}}^{rad} \\ &= -\frac{1}{e^2} \int_{\mathcal{N}} \left[i (L_{\ell}(\mathbf{d}\alpha) + (\frac{1}{2}\vartheta_{(\ell)} - i\varphi_{(\ell)}) \, \mathbf{d}\alpha) \, \mathbf{d}\bar{\alpha} \, k \wedge m \wedge \bar{m} + \text{cc.} \right]. \end{aligned}$$

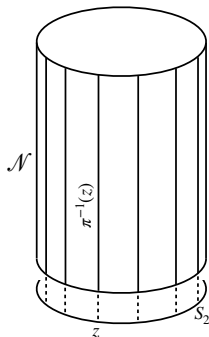
Signature (0++) metric.

$$q_{ab} = \delta_{ij} e^i_a e^j_b, \quad i, j = 1, 2.$$

Parametrisation of the dyad

$$e^i = \Omega S^i_m e^m_{(o)}.$$

- **Conformal factor** Ω parametrizes the overall scale.
- **$SL(2, \mathbb{R})$ -Holonomy** S^i_m determines the shape degrees of freedom.
- Fiducial background dyad $e^j_{(o)}$, e.g. $(e^0_{(o)}, e^1_{(o)}) = (d\vartheta, \sin \vartheta d\varphi)$.
- Clock variable u on \mathcal{N} such that $\ell[u] = 1, \ell[\vartheta] = \ell[\varphi] = 0$.
- Null surface has upper (lower) boundary: $u|_{\mathcal{E}_\pm} = \pm 1$.



To simplify the canonical analysis, we consider

$$\begin{aligned} \text{expansion:} \quad & \vartheta_{(\ell)} = \Omega^{-1} \frac{d}{du} \Omega, \\ U(1) \text{ holonomy:} \quad & e^{-i\Delta} = e^{-i \int_{-1}^u du' \varphi_{(\ell)}(u', \vartheta, \varphi)}, \\ \text{fiducial volume form:} \quad & d^3 v_o = \sin \vartheta \, du \wedge d\vartheta \wedge d\varphi. \end{aligned}$$

Then, we simply have

$$\Omega_{\mathcal{N}}^{rad} = -\frac{1}{e^2} \int_{\mathcal{N}} d^3 v_o \left[\frac{d}{du} \left(\Omega e^{-i\Delta} d\alpha \right) \Omega e^{i\Delta} d\bar{\alpha} + \text{cc.} \right],$$

Resulting Poisson brackets (with skew-symmetric Θ -step function)

$$\left\{ \alpha(u_1, \vartheta_1, \varphi_1), \bar{\alpha}(u_2, \vartheta_2, \varphi_2) \right\} = \frac{e^2}{2} \Theta(u_2 - u_1) \Omega_1^{-1} \Omega_2^{-1} e^{+i(\Delta_1 - \Delta_2)} \delta_{12}^{(2)}.$$

Side remark: Mode expansion

Toy model: ignore angular Y_{lm} -modes, consider symplectic two-form

$$\Omega = -\frac{1}{e^2} \int_{-1}^1 du \frac{d}{du} \mathbb{d}\alpha(u) \mathbb{d}\bar{\alpha}(u) + \text{cc.}$$

Split canonical variables

$$\alpha(u) = \alpha_{\text{soft}}(u) + \alpha_{\text{hard}}(u).$$

We introduce soft parts and hard parts

$$\alpha_{\text{soft}}(u) = \frac{1}{2} \Delta \alpha u,$$

$$\alpha_{\text{hard}}(u) = \sum_{n=-\infty}^{\infty} \alpha_n e^{-in\pi u}.$$

Fundamental Poisson brackets

$$\{p_{\Delta\alpha}, \Delta\alpha\} = \{p_{\Delta\bar{\alpha}}, \Delta\bar{\alpha}\} = 1,$$

$$\{\alpha_n, \bar{\alpha}_m\} = -\frac{i e^2}{4\pi} \frac{1}{n} \delta_{n-m}, \quad n, m \neq 0,$$

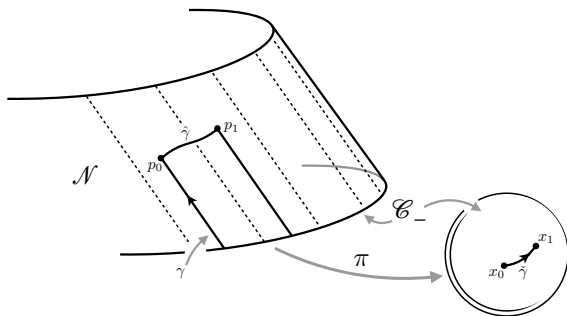
where $p_{\Delta\alpha}$ is conjugate to the finite memory $\Delta\alpha = \alpha(1) - \alpha(-1)$,

$$p_{\Delta\alpha} = \frac{1}{e^2} \left(\bar{\alpha}_0 + \sum_{n \neq 0} (-1)^{n+1} \bar{\alpha}_n \right).$$

Holonomy algebra

Wilson lines on null surfaces

We consider Wilson lines along a path γ (hoop) that consist of two null segments connected by a spacelike path $\hat{\gamma}$.

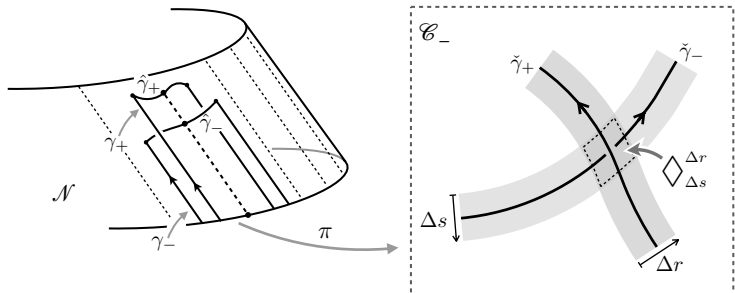


Wilson line operators

$$h_\gamma[A] = \exp\left(-i \int_{\hat{\gamma}} (\alpha \bar{m} + \bar{\alpha} m)\right) e^{-i((\lambda \circ \gamma)(1) - (\lambda \circ \gamma)(0))}.$$

Regularised holonomies

Poisson brackets among holonomies vanish unless Wilson lines intersect the same light ray.



Regularised holonomies

$$h_{\hat{\gamma}_{\pm}} = \lim_{\Delta\epsilon \searrow 0} \exp\left(-i \frac{1}{\Delta\epsilon} \int_{\tilde{\gamma}_{\pm}^{\Delta\epsilon}} d\epsilon \wedge \eta_{\pm}^* (\alpha \bar{m} + \bar{\alpha} m)\right).$$

For vanishing shear, algebra simply given by

$$\{h_{\gamma_+}, h_{\gamma_-}\} = \frac{e^2}{4} h_{\gamma_+} h_{\gamma_-} \frac{\cos \varphi(\gamma_+, \gamma_-)}{|\sin \varphi(\gamma_+, \gamma_-)|},$$

where $\varphi(\gamma_+, \gamma_-)$ is the angle at the intersection in the projection.

For non-vanishing shear, algebra depends on parallel transport

$$\{h_{\gamma_+}, h_{\gamma_-}\} = \frac{e^2}{4} h_{\gamma_+} h_{\gamma_-} \frac{\check{v}_+^a \check{v}_-^b H_a^c(p_+) H_b^d(p_-) q_{cd}}{|\varepsilon_{fg} \check{v}_+^f \check{v}_-^g|} =: \frac{e^2}{4} g(\gamma_+, \gamma_-) h_{\gamma_+} h_{\gamma_-}.$$

where we introduced the holonomy (gravitational memory)

$$H_{\underline{a}}^{\underline{b}}(p_{\pm}) = \text{Pexp} \left(\int_{-1}^{u_{\pm}} du \sigma(u, z, \bar{z}) \right)_{\underline{a}}^{\underline{b}},$$

which depends on the shear tensor $\sigma_{\underline{a}}^{\underline{b}}$, in here $\underline{a}, \underline{b}, \dots$ are tensors indices on $T\mathcal{C}_-$ and $\check{v}_{\pm}^{\underline{a}} = \pi_* \dot{\gamma}_{\pm}^{\underline{a}}$ at the intersection.

Dirac bracket for $SL(2, \mathbb{R})$ holonomy (perhaps mnemonomy better name)

$$\left\{ S_m^i(x), S_n^j(y) \right\}^* = -4\pi G \Theta(x, y) \delta^{(2)}(x, y) \Omega^{-1}(x) \Omega^{-1}(y) \\ \times \left[e^{-2i(\Delta(x) - \Delta(y))} [XS(x)]_m^i [\bar{X}S(y)]_n^j + \text{cc.} \right].$$

$U(1)$ angle: $\Delta = \int^{\mathcal{U}} d\mathcal{U} \varphi(\mathcal{U})$, $\varphi = -\frac{1}{2} \text{Tr} \left(J \frac{d}{d\mathcal{U}} S S^{-1} \right)$.

Gauge symmetries:

- 1 $U(1)$ transformations
- 2 vertical diffeomorphisms along null generators

Quantum holonomy algebra

Classical algebra: Fix a graph, i.e. collection of hoops $\Gamma = (\gamma_1, \gamma_2, \dots, \gamma_N)$, we obtain

$$\{h_{\gamma_i}, h_{\gamma_j}\} = \frac{e^2}{4} g(\gamma_i, \gamma_j) h_{\gamma_i} h_{\gamma_j},$$

note: $g(\gamma_i, \gamma_j)$ singular as $\gamma_i \rightarrow \gamma_j$.

Quantisation:

$$[\hat{h}_{\gamma_i}, \hat{h}_{\gamma_j}] = \frac{\hbar e^2}{8i} g(\gamma_i, \gamma_j) (\hat{h}_{\gamma_i} \hat{h}_{\gamma_j} + \hat{h}_{\gamma_j} \hat{h}_{\gamma_i}),$$

in other words:

$$\hat{h}_{\gamma_i} \hat{h}_{\gamma_j} = e^{-2\pi i q_{ij}} \hat{h}_{\gamma_j} \hat{h}_{\gamma_i}.$$

note: $g(\gamma_i, \gamma_j)$ regular as $\gamma_i \rightarrow \gamma_j$, implying

$$\hat{h}_{\gamma_i} \hat{h}_{\gamma_i} = 0.$$

Finally, we have to impose reality conditions:

Classical level:

$$h_\gamma \bar{h}_\gamma = 1.$$

Quantum level:

$$h_\gamma h_\gamma^\dagger + h_\gamma^\dagger h_\gamma = 2\mathbb{1}.$$

Summary of algebraic relations:

$$\begin{aligned}\hat{h}_{\gamma_i} \hat{h}_{\gamma_j} &= e^{-2\pi i q_{ij}} \hat{h}_{\gamma_j} \hat{h}_{\gamma_i}, \\ \hat{h}_{\gamma_i} \hat{h}_{\gamma_j}^\dagger &= e^{+2\pi i q_{ij}} \hat{h}_{\gamma_j}^\dagger \hat{h}_{\gamma_i} + 2\delta_{ij} \mathbb{1}.\end{aligned}$$

Finally, we have to impose reality conditions:

For a single link

$$\begin{aligned}\hat{h}_\gamma^\dagger|\gamma, \uparrow\rangle &= 0, & \hat{h}_\gamma|\gamma, \uparrow\rangle &= \sqrt{2}|\gamma, \downarrow\rangle, \\ \hat{h}_\gamma^\dagger|\gamma, \downarrow\rangle &= \sqrt{2}|\gamma, \uparrow\rangle, & \hat{h}_\gamma|\gamma, \downarrow\rangle &= 0.\end{aligned}$$

The reality conditions $h_{\gamma^{-1}} = h_\gamma^\dagger$ imply

$$|\gamma^{-1}, \uparrow\rangle = |\gamma, \downarrow\rangle.$$

To extend this construction to the entire holonomy algebra, we assign to every oriented graph $\gamma \in \Gamma$ an element of $\{\uparrow, \downarrow\}$, i.e. a framing $s(\gamma)$.

Vacuum state

$$\hat{h}_{\gamma_{\pi(1)}}^{s(\gamma_{\pi(1)})} \cdots \hat{h}_{\gamma_{\pi(K)}}^{s(\gamma_{\pi(K)})} |s, \Gamma\rangle = 0, \quad \text{where} \quad \begin{cases} \hat{h}_{\gamma}^{\uparrow} = \hat{h}_{\gamma}^{\dagger}, \\ \hat{h}_{\gamma}^{\downarrow} = \hat{h}_{\gamma}, \end{cases}$$

for all subsequences $(\gamma_{\pi(1)}, \dots, \gamma_{\pi(K)})$ of $(\gamma_1, \dots, \gamma_N)$.

In this way, we obtain a 2^N -dimensional Hilbertspace \mathcal{H}_{Γ} .

To realise path refinements, e.g. $\gamma \rightarrow (\gamma, \alpha, \beta)$, $\gamma = \beta \circ \alpha$, we impose constraints $C = h_{\gamma} - h_{\alpha} h_{\beta} = 0$ on extended Hilbert space $\mathcal{H}_{(\gamma)} \otimes \mathcal{H}_{(\gamma, \alpha, \beta)}$.

Conclusion

- On null surface \mathcal{N} , Wilson lines no longer commute.
- Direct consequence of existence of (anti-)selfdual two-forms $k \wedge m$ ($k \wedge \bar{m}$) on \mathcal{N} , implying that magnetic and electric fluxes are no longer independent initial data.
- Wilson lines that intersect the same light ray no longer commute.
- We developed a quantisation of the resulting holonomy algebra. Each holonomy becomes an anti-commuting number.
- No preferred vacuum. Vacuum state depends on choice of framing.