$SL(2,\mathbb{R})$ Holonomies on the Light Cone

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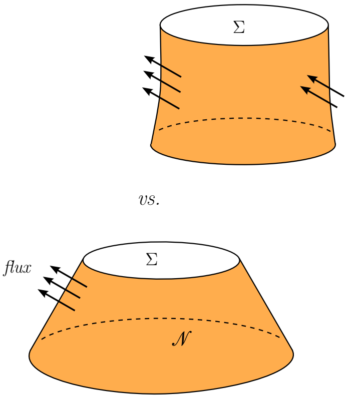
Introduction (one slide)

Subsystems as evolving regions in space

To characterise a gravitational subsystem,
two choices must be made.

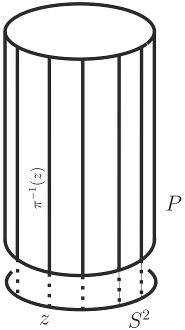
- A choice must be made for how to extend the boundary of the partial Cauchy hypersurface Σ into a worldtube $\mathscr{N}$.
- A choice must be made for what is the flux of gravitational radiation across the worldtube of the boundary, i.e. a (background field, c-number) that drives the time-dependence of the Hamiltonian.



N.B.: In spacetime dimensions $d < 4$, there are no gravitational waves, and we can forget about the second issue. The Hamiltonian will be automatically conserved.

Covariant phase space, Holst action, causal regions

- Compact spacetime region $\mathscr{M}$.
- Bounded by spacelike disks M_0, M_1 and null surface $\mathscr{N}$.
- Null surface boundary $\mathscr{N}$ embedded into abstract bundle (ruled surface) $P(\pi, \mathscr{C}) \simeq \mathbb{R} \times \mathscr{C}$.
- Null generators $\pi^{-1}(z)$.



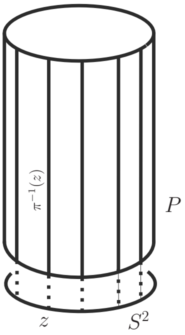
Bulk plus boundary configuration variables

Fields in the interior of spacetime:

- Soldering form (tetrads): $e_{AA'}$.
- Self-dual two-forms:
  $e_{AA'} \wedge e_{BB'} = -\epsilon_{AB}\bar{\Sigma}_{A'B'} + \mathrm{cc}.$
- Spin connection:
  $\nabla\psi^A = \mathrm{d} \wedge \psi^A + A^A{}_B \wedge \psi^B.$

Fields at the boundary of spacetime:

- Null flag ℓ^A: $l^a \simeq \mathrm{i}\ell^A\bar{\ell}^{A'}$.
- Conjugate spinor-valued two-form
  $\eta_A \in \Omega^2(\mathcal{N} : \mathcal{S})$: $\varphi^*_{\mathcal{N}}\Sigma_{AB} = \eta_{(A}\ell_{B)}$.
- Area two-form:
  $\varepsilon = \mathrm{i}\eta_A\ell^A \in \Omega^2(\mathcal{N} : \mathbb{R})$.
- Abelian Ashtekar–Barbero
  connection $\varkappa \in \Omega^1(\mathcal{N} : \mathbb{R})$.



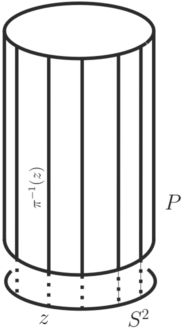
Co-basis at the boundary

Adapted co-basis $(k_a, m_a, \bar{m}_a)$:

- given the metric in the interior, co-dyads $(m_a, \bar{m}_a) \in \Omega^1(\mathscr{N} : \mathbb{C})$ are unique modulo $U(1)$ symmetry: $m_a \longrightarrow \mathrm{e}^{\mathrm{i}\varphi} m_a$.
- co-vector k_a is unique modulo Lorentz trafos $k_a \longrightarrow \mathrm{e}^{-f} k_a + \zeta \bar{m}_a + \zeta m_a$.
- dual null vector $l^a \in T\mathscr{N} : k_a l^a = -1, \pi_* l^a = 0$.

Associate spin dyad (k_A, ℓ_A):

- Normalized: $k_A \ell^A = 1$.
- $\eta_A = (\ell_A\, k - k_A m) \wedge \bar{m} \in \Omega^2(\mathscr{N} : \mathscr{S})$



Bulk plus boundary action:

$$S[A,e|k,\ell|\varkappa,k,m,\bar{m}] = \frac{\mathrm{i}}{8\pi\gamma G}(\gamma+\mathrm{i})\Big[\int_{\mathscr{M}}\Big(\Sigma_{AB}\wedge F^{AB} - \frac{\Lambda}{6}\Sigma_{AB}\wedge\Sigma^{AB}\Big) + \int_{\mathscr{N}}\eta_A\wedge\big(D-\tfrac{1}{2}\varkappa\big)\ell^A\Big] + \mathrm{cc.}$$

Boundary conditions along $\mathscr{N}$: $\delta[\varkappa_a, l^a, m_a]/_\sim = 0$

- vertical diffeomorphisms $[\varphi^*\varkappa_a, l^a, \varphi^* m_a] \sim [\varkappa_a, \varphi_* l^a, m_a]$
- dilations $[\varkappa_a, l^a, m_a] \sim [\varkappa_a + \nabla_a f, \mathrm{e}^f l^a, m_a]$
- complexified conformal transformations $\lambda = \mu + \mathrm{i}\nu$:
  $[\varkappa_a, l^a, m_a] \sim \Big[\varkappa_a - \frac{1}{\gamma}\nabla_a\nu, \mathrm{e}^\mu \ell^A, \mathrm{e}^{\mu+\mathrm{i}\nu} m_a\Big]$
- shifts $[\varkappa_a, l^a, m_a] \sim [\varkappa_a + \bar{\zeta} m_a + \zeta\bar{m}_a, l^a, m_a]$

The equivalence class $g = [\varkappa_a, l^a, m_a]/_\sim$ characterises two degrees of freedom per point.

Covariant symplectic potential

Symplectic potential:

$$\boxed{\Theta_{\mathcal{N}} = -\frac{1}{8\pi G}\int_{\mathcal{N}} \varepsilon \wedge \mathrm{d}\varkappa + \frac{\mathrm{i}}{8\pi\gamma G}\int_{\mathcal{N}} \Big((\gamma + \mathrm{i})\ell_A D\ell^A \wedge \mathrm{d}(k \wedge \bar{m}) - \mathrm{cc.}\Big)}$$

Area two-form: $\varepsilon = -\mathrm{i} m \wedge \bar{m}$.

Shear and expansion:

$$\ell_A D\ell^A = -\left(\frac{1}{2}\vartheta_{(l)} m + \sigma_{(l)}\bar{m}\right)$$

Gauge symmetries:

- vertical diffeomorphisms $\delta_\xi^{diff}[\cdot] = \mathscr{L}_\xi[\cdot] : \xi^a \sim l^a \in T\mathcal{N}$
- $U(1)$ transformations $\delta_\varphi^{U(1)}[\varkappa_a, m_a] = [-\gamma^{-1}\partial_a\varphi, \mathrm{i}\,\varphi m_a]$
- dilations $\delta_f^{dilat}[\varkappa_a, l^a] = [\partial_a f, f l^a]$
- shift symmetry $\delta_\zeta^{shift}[\varkappa_a] = \zeta\bar{m}_a + \bar{\zeta} m_a$

Transition to $SL(2,\mathbb{R})$ variables

Covariant vs. kinematical phase space

In gravity, covariant phase-space methods are useful to

- identify gauge symmetries,
- calculate charges,
- derive the first-law of BH thermodynamics.

Less useful to identify Dirac observables and their algebra.

Strategy ahead:

1. embed covariant phase space into larger kinematical phase space.
2. impose constraints that bring us down to physical phase space.

Auxiliary two-dimensional vector space $\mathbb{V}$ with complex basis $(m^i, \bar{m}^i)$, $i = 0, 1$, and internal metric q_{ij}, $q^{ij} : q^{ik} q_{kj} = \delta^i_j$.

Fiducial dyad

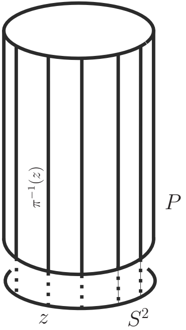
$$e^i_{(o)} = \bar{m}^i \frac{\mathrm{d}z}{1 + |z|^2} + \mathrm{cc.},$$
$$\delta[e^i_{(o)}] = 0.$$

Fiducial area

$$e^i_{(o)} \wedge e^j_{(o)} = \varepsilon^{ij} d^2 v_o.$$

Parametrisation of the dyad

$$\boxed{e^i = \Omega\, S^i{}_j\, e^j_{(o)}.}$$



Basic variables are now: $S^i{}_j : \mathcal{N} \to SL(2, \mathbb{R})$ and conformal factor $\Omega : \mathcal{N} \to \mathbb{R}$.

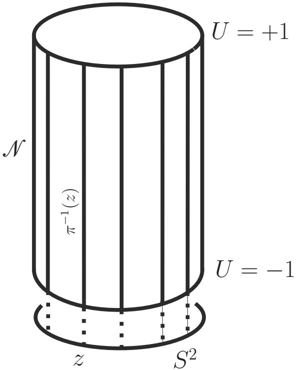
Convenient time variable $U : \mathscr{N} \to \mathbb{R}$, such that

Boundary condition at $\partial\mathscr{N} = \mathscr{C}_+ \cup \mathscr{C}_-$,

$$U(\partial\mathscr{N}, z, \bar{z}) = \pm 1,$$

Non-affinity equals expansion

$$\partial_U^b \nabla_b \partial_U^a = -\frac{1}{2}(\Omega^{-2}\frac{\mathrm{d}}{\mathrm{d}U}\Omega^2)\partial_U^a$$



Nota bene: $\delta U \neq 0$, but $\delta U\big|_{\partial\mathscr{N}} = 0$.

Step 2: Symplectic potential

Quantities with a circumflex are pull-backs to the fibres $\gamma_z = \pi^{-1}(z)$.

$$\Theta_{\mathcal{N}} = \frac{1}{8\pi G}\int_{\mathcal{N}} d^2 v_o \wedge \Big[p_K \mathbb{d}\widetilde{K} + \gamma^{-1} E\, \mathbb{d}\widetilde{\Phi} + \widetilde{\Pi}^i{}_j \left[S \mathbb{d} S^{-1}\right]^j{}_i\Big] + \textit{corner term.}$$

Abelian variables:

$U(1)$ angle: $\widetilde{\Phi} := -\varphi_{(l)}\varphi^*_{\gamma_z} k$, area: $E := \Omega^2$, lapse: $\widetilde{K} := đU \equiv \varphi^*_{\gamma_z} \mathrm{d}U$.

$SL(2,\mathbb{R})$ holonomy flux variables

$$\begin{aligned}
\{\widetilde{\Pi}(x), S(y)\} &= -8\pi G\, X S(y)\, \tilde{\delta}_{\mathcal{N}}(x,y),\\
\{\widetilde{I}(x), S(y)\} &= +4\pi G\, J S(y)\, \tilde{\delta}_{\mathcal{N}}(x,y),\\
\{\widetilde{\Pi}(x), \widetilde{I}(y)\} &= -8\pi \mathrm{i} G\, \widetilde{\Pi}(y)\, \tilde{\delta}_{\mathcal{N}}(x,y),\\
\{\widetilde{\Pi}(x), \bar{\widetilde{\Pi}}(y)\} &= -16\pi \mathrm{i} G\, \widetilde{I}(y)\, \tilde{\delta}_{\mathcal{N}}(x,y),
\end{aligned}$$

Basis in $SL(2,\mathbb{R})$ such that $\widetilde{\Pi}^i{}_j = \widetilde{I} J^i{}_j + \widetilde{\Pi} \bar{X}^i{}_j + \bar{\widetilde{\Pi}} X^i{}_j$,
and $[J, X] = -2\mathrm{i}X$, $[X, \bar{X}] = \mathrm{i}J$.

First-class constraints

$U(1)$ Gauss constraint

$$\forall\Lambda : G[\Lambda] = \int_{\mathscr{N}} d^2 v_o \wedge \Lambda \left(\widetilde{I} - \frac{1}{2\gamma}\, đ\, E\right) \stackrel{!}{=} 0,$$

Hamilton constraint/Raychaudhuri equation

$$\forall\xi^a : \pi_*\xi^a = 0 : H_\xi = -\frac{1}{4\pi G}\int_{\mathscr{N}} d^2 v_o \wedge \mathrm{d}U\, \mathscr{L}_\xi[U]\left[\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}U^2}\Omega^2 + \sigma\bar{\sigma}\right] \stackrel{!}{=} 0,$$

Shear in terms of the off-diagonal components of $\mathfrak{sl}(2,\mathbb{R})$-valued momentum

$$\widetilde{\Pi} := \frac{\gamma + \mathrm{i}}{\gamma}\Omega\,\sigma\, đU$$

Define $\mathfrak{sl}(2,\mathbb{R})$ connection

$$\text{đ}\, S \cdot S^{-1} =: \widetilde{\varphi} J + \widetilde{h} \bar{X} + \widetilde{\bar{h}} X,$$

Second-class constraints

$$\forall \mu : D[\mu] = \int_{\mathcal{N}} d^2 v_o \wedge \mu \left(\tilde{\Phi} - \tilde{\varphi} \right) \stackrel{!}{=} 0,$$

$$\forall \zeta : V[\bar{\zeta}] = \int_{\mathcal{N}} d^2 v_o \wedge \bar{\zeta} \mathrm{e}^{-2\mathrm{i}\Delta} \left(\Omega^{-1} \widetilde{\Pi} - \frac{\gamma + \mathrm{i}}{\gamma} \Omega \widetilde{h} \right) \stackrel{!}{=} 0,$$

$$\forall \lambda : C[\lambda] = \int_{\mathcal{N}} d^2 v_o \wedge \lambda \left(p_K \widetilde{K} - \text{đ}\, E \right) \stackrel{!}{=} 0,$$

$U(1)$ connection

$$\Delta(u, z, \bar{z}) = \int_{\gamma_z(u)} \widetilde{\varphi},$$

Dirac bracket for $SL(2,\mathbb{R})$ holonomy

$$\begin{aligned}\left\{S^i{}_m(x), S^j{}_n(y)\right\}^* = -4\pi G\,\Theta(x,y)\,\delta^{(2)}(x,y)\,\Omega^{-1}(x)\,\Omega^{-1}(y)\\ \times\left[\mathrm{e}^{-2\,\mathrm{i}\,(\Delta(x)-\Delta(y))}\left[X S(x)\right]^i{}_m\left[\bar{X} S(y)\right]^j{}_n + \mathrm{cc.}\right].\end{aligned}$$

Dirac observables can be constructed using standard techniques.

Gauge symmetries:

1. $U(1)$ transformations
2. vertical diffeomorphisms along null generators

Summary

Summary

- Action with Barbero–Immirzi parameter γ in causal regions
- γ mixes $U(1)$ frame rotations and dilations
- Kinematical phase space carries $SL(2,\mathbb{R})$ holonomy-flux algebra
- All constraints are polynomial in the fundamental fields