$SL(2,\mathbb{R})$ Holonomies on the Light Cone

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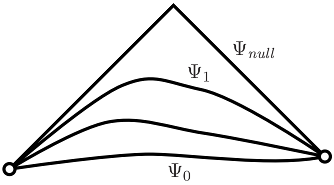
Introduction

Why quantum gravity in causal regions? Different views:

- Mere gauge fixing: Represent diffeomorphism equivalence class of states $[\Psi_0]$ by states on the light cone.
- Coarse graining: Build observables by successively gluing gravitational subsystems.



- Soft modes/edge modes: In gravity, energy, momentum, angular momentum, center of mass, supertranslations ... are analogous to charge in QED. Do we have superpositions of such charges in nature? Can we build them in the lab? Can they help us understand black hole information loss?

[Strominger, Perry; Godazgar, Harlow, Wu; Kartik, Chandrasekaran, Flanagan, Bonga; Freidel, Donnelly, Speranza, Riello, Geiller, Livine, Dittrich, Pranzetti, ww; Grumiller, Seraj, Barnich, Compère,...]

Gravity in terms of differential forms

To understand how gravity couples to boundaries, it is useful to work with differential forms rather than tensors since there is a natural notion of projection onto the boundary, namely the pull-back $\varphi^* : T^*M \to T^*(\partial M)$, which does not require a metric.

Fundamental configuration variables

$$
\begin{aligned}
g_{ab} &= \eta_{\alpha\beta} e^{\alpha}{}_{a} e^{\beta}{}_{b}, \\
\nabla \wedge T^{\alpha\dots}{}_{\beta\dots} &= \mathrm{d} \wedge T^{\alpha\dots}{}_{\beta\dots} + A^{\alpha}{}_{\mu} \wedge T^{\mu\dots}{}_{\beta\dots} + \dots \\
&\quad - A^{\mu}{}_{\beta} \wedge T^{\alpha\dots}{}_{\mu\dots} - \dots .
\end{aligned}
$$

Palatini action

$$S[A,e] = \frac{1}{16\pi G} \int_{\mathscr{M}} \underbrace{*(e_\alpha \wedge e_\beta)}_{\Sigma_{\alpha\beta}} \wedge F^{\alpha\beta}[A] + \text{boundary terms.}$$

Symplectic potential

$$\Theta_\Sigma = \frac{1}{16\pi G} \int_\Sigma \underbrace{*\Sigma_{\alpha\beta} \wedge \mathbb{d} A^{\alpha\beta}}_{\text{“}p\,\mathrm{d}x\text{”}} + \text{corner terms.}$$

Two kinds of gauge symmetries: diffeomorphisms and internal Lorentz transformations.

Lorentz transformations

$$\delta_\Lambda[e^\alpha] = \Lambda^\alpha{}_\beta e^\beta, \qquad \Lambda_{\alpha\beta} = -\Lambda_{\beta\alpha}$$
$$\delta_\Lambda[A^\alpha{}_\beta] = -\nabla\Lambda^\alpha{}_\beta.$$

Lorentz charges are integrable at full non-perturbative level.

$$\Omega_\Sigma(\delta_\Lambda,\delta)\big|_{\mathrm{EOM}} = -\delta[Q_\Lambda].$$
$$Q_\Lambda[\Sigma] = -\frac{1}{16\pi G}\oint_{\partial\Sigma} *\Sigma_{\alpha\beta}\Lambda^{\alpha\beta}.$$

NB: Such Lorentz charges do not exist in metric gravity (on the ADM phase space). Physically meaningful perhaps only if we add fermions (defects of torsion).

Charges and symmetries 2/2

Two kinds of gauge symmetries: diffeomorphisms and internal Lorentz transformations.

Base diffeomorphisms lifted upwards into the Lorentz bundle

$$\delta_\xi[e^\alpha] = \nabla(\xi \lrcorner e^\alpha) + \xi \lrcorner (\nabla \wedge e^\alpha),$$
$$\delta_\xi[A^\alpha{}_\beta] = \xi \lrcorner F^\alpha{}_\beta\,.$$

Diffeomorphism charges

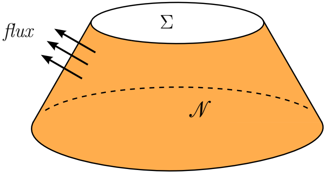
$$\Omega_\Sigma(\delta_\xi, \delta)\big|_{\mathrm{EOM}} = \frac{1}{16\pi G}\oint_{\partial\Sigma} \xi \lrcorner * \Sigma_{\alpha\beta} \wedge \delta A^{\alpha\beta} \stackrel{?}{=} -\delta[P_\xi].$$

Trivially integrable at linear order in perturbations

$$e^\alpha = \mathring{e}^\alpha + f^\alpha \equiv \mathring{e}^\alpha + f^\alpha{}_\beta\, \mathring{e}^\beta, \quad f_{\alpha\beta} = f_{\beta\alpha},$$
$$P_\xi = \frac{1}{8\pi G}\oint_{\partial\Sigma} \xi \lrcorner * \mathring{\Sigma}_{\alpha\beta} \wedge \mathring{\nabla}^{[\alpha} f^{\beta]}.$$

NB: for an asymptotic time translation $\xi^a = \left[\frac{\partial}{\partial x^0}\right]^a$, the linearised charge P_ξ returns the ADM mass for a linearised solution $f_{\alpha\beta} = \mathcal{O}(r^{-1})$ around $\mathring{e}^\alpha = \mathrm{d}x^\alpha$.

- In gravity, time evolution $t \to t + \varepsilon$ can be understood as a large gauge transformation.
- It seems reasonable to expect the Hamiltonian is the generator for such a gauge transformation:



$$H[\Sigma] \equiv P_\xi[\Sigma] \overset{?}{=} \oint_{\partial\Sigma} d^2v^a\, \xi^b T_{ab}[?].$$

- We assume that P_ξ generates the symmetry algebra

$$\{P_\xi, P_{\xi'}\} = -P_{[\xi,\xi']} + c[\xi,\xi'].$$

- However, that's at odds with the fact that a system may loose mass via gravitational radiation

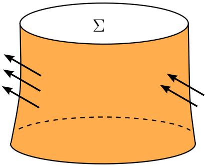
$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Mc^2 = \frac{\mathrm{d}}{\mathrm{d}t}H &= \{H, H\} = 0,\\ &= -\frac{1}{4\pi G}\oint_{S^2_t} d^2\Omega\, |\dot{\sigma}^0|^2 \leq 0.\end{aligned}\right\} \lightning$$

- ... unless, we allow for an explicit time dependence in the Hamiltonian ...

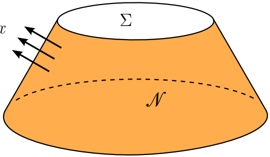
Subsystems as evolving regions in space

To characterise a gravitational subsystem,
two choices must be made.

- A choice must be made for how to extend the boundary of the partial Cauchy hypersurface Σ into a worldtube $\mathscr{N}$.
- A choice must be made for what is the flux of gravitational radiation across the worldtube of the boundary, i.e. a (background field, c-number) that drives the time-dependence of the Hamiltonian.



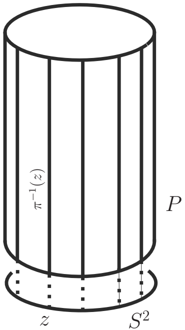
vs.



N.B.: In spacetime dimensions $d < 4$, there are no gravitational waves, and we can forget about the second issue. The Hamiltonian will be automatically conserved.

Covariant phase space, Holst action, causal regions

- Compact spacetime region $\mathscr{M}$.
- Bounded by spacelike disks M_0, M_1 and null surface $\mathscr{N}$.
- Null surface boundary $\mathscr{N}$ embedded into abstract bundle (ruled surface) $P(\pi, \mathscr{C}) \simeq \mathbb{R} \times \mathscr{C}$.
- Null generators $\pi^{-1}(z)$.



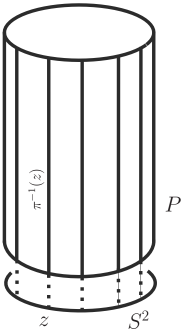
Bulk plus boundary configuration variables

Fields in the interior of spacetime:

- Soldering form (tetrads): $e_{AA'}$.
- Self-dual two-forms:
  $e_{AA'} \wedge e_{BB'} = -\epsilon_{AB}\bar{\Sigma}_{A'B'} + \mathrm{cc}.$
- Spin connection acting on e.g. spinor fields:
  $\nabla\psi^A = \mathrm{d} \wedge \psi^A + A^A{}_B \wedge \psi^B.$

Fields at the boundary of spacetime:

- Null flag ℓ^A: $l^a \simeq \mathrm{i}\ell^A\bar{\ell}^{A'}$.
- Conjugate spinor-valued two-form $\eta_A \in \Omega^2(\mathcal{N} : \mathcal{S})$: $\varphi^*_{\mathcal{N}}\Sigma_{AB} = \eta_{(A}\ell_{B)}$.
- Area two-form:
  $\varepsilon = \mathrm{i}\eta_A\ell^A \in \Omega^2(\mathcal{N} : \mathbb{R})$.
- Abelian Ashtekar–Barbero connection $\varkappa \in \Omega^1(\mathcal{N} : \mathbb{R})$.

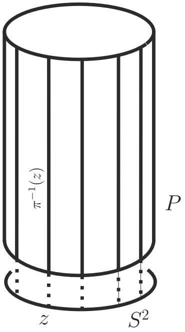


Adapted co-basis $(k_a, m_a, \bar{m}_a)$:

- given the metric in the interior, co-dyads $(m_a, \bar{m}_a) \in \Omega^1(\mathcal{N} : \mathbb{C})$ are unique modulo $U(1)$ symmetry: $m_a \longrightarrow \mathrm{e}^{\mathrm{i}\varphi} m_a$.
- co-vector k_a is unique modulo Lorentz trafos $k_a \longrightarrow \mathrm{e}^{-f} k_a + \zeta \bar{m}_a + \zeta m_a$.
- dual null vector $l^a \in T\mathcal{N} : k_a l^a = -1, \pi_* l^a = 0$.

Associate spin dyad (k_A, ℓ_A):

- Normalized: $k_A \ell^A = 1$.
- $\eta_A = (\ell_A\, k - k_A m) \wedge \bar{m} \in \Omega^2(\mathcal{N} : \mathcal{S})$



Bulk plus boundary action:

$$\boxed{\begin{aligned}S[A,e|k,\ell|\varkappa,k,m,\bar{m}] = \frac{\mathrm{i}}{8\pi\gamma G}(\gamma+\mathrm{i})\Big[\int_{\mathscr{M}}\Big(\Sigma_{AB}\wedge F^{AB}-\frac{\Lambda}{6}\Sigma_{AB}\wedge\Sigma^{AB}\Big)+\\ +\int_{\mathscr{N}}\eta_A\wedge\big(D-\tfrac{1}{2}\varkappa\big)\ell^A\Big]+\text{cc.}\end{aligned}}$$

Boundary conditions along $\mathscr{N}$: $\delta[\varkappa_a, l^a, m_a]/_\sim = 0$

- vertical diffeomorphisms $[\varphi^*\varkappa_a, l^a, \varphi^* m_a] \sim [\varkappa_a, \varphi_* l^a, m_a]$
- dilations $[\varkappa_a, l^a, m_a] \sim [\varkappa_a + \nabla_a f, \mathrm{e}^f l^a, m_a]$
- complexified conformal transformations $\lambda = \mu + \mathrm{i}\nu$: $[\varkappa_a, l^a, m_a] \sim \Big[\varkappa_a - \frac{1}{\gamma}\nabla_a\nu, \mathrm{e}^\mu \ell^A, \mathrm{e}^{\mu+\mathrm{i}\nu} m_a\Big]$
- shifts $[\varkappa_a, l^a, m_a] \sim [\varkappa_a + \bar{\zeta} m_a + \zeta\bar{m}_a, l^a, m_a]$

The equivalence class $g = [\varkappa_a, l^a, m_a]/_\sim$ characterises two degrees of freedom per point.

Covariant pre-symplectic potential for the partial Cauchy surfaces:

$$\Theta_\Sigma = \frac{\mathrm{i}}{8\pi\gamma G}(\gamma + \mathrm{i}) \left[-\oint_{\mathscr{C}} \eta_A \mathrm{d}\ell^A + \int_\Sigma \Sigma_{AB} \wedge \mathrm{d}A^{AB} \right] + \mathrm{cc.}$$

Phase space of bulk and boundary degrees of freedom:

$$P_{phys} = (P_{bulk} \times P_{bndry})/_{gauge}$$

Poisson brackets at the two-dimensional corner

$$\big\{\pi_A(z), \ell^B(z')\big\}_{\mathscr{C}} = \delta^B_A \delta^{(2)}(z, z').$$

Canonical (spinor-valued) momentum

$$\pi_A = \frac{\mathrm{i}}{8\pi G} \frac{\gamma + \mathrm{i}}{\gamma} \eta_A.$$

Area operator

- The cross-sectional oriented area is

$$\boxed{\mathrm{Area}[\mathscr{C}] = -8\pi G \frac{\mathrm{i}\gamma}{\gamma+\mathrm{i}} \oint_{\mathscr{C}} d^2x\, \pi_A \ell^A.}$$

- For the area to be real-valued (charge neutral), we have to satisfy the reality conditions,

$$\boxed{K - \gamma L = 0.}$$

- Generators of complexified $U(1)_{\mathbb{C}}$ transformations

$$L = -\frac{1}{2\mathrm{i}}\pi_A \ell^A + \mathrm{cc.} \quad \textit{(generator of U(1) transformations)},$$
$$K = -\frac{1}{2}\pi_A \ell^A + \mathrm{cc.} \quad \textit{(dilatations of the light like direction).}$$

- Generators of generalised angular moments

$$J_\xi[\mathscr{C}] = \oint_{\mathscr{C}} \left(\pi_A \xi^a D_a \ell^A + \mathrm{cc.}\right), \quad \xi^a \in T\mathscr{C}.$$

Phase space on the null surface: radiative modes

Symplectic potential:

$$\boxed{\Theta_{\mathscr{N}} = -\frac{1}{8\pi G}\int_{\mathscr{N}} \varepsilon \wedge \mathbb{d}\varkappa + \frac{\mathrm{i}}{8\pi\gamma G}\int_{\mathscr{N}}\Big((\gamma+\mathrm{i})\ell_A D\ell^A \wedge \mathbb{d}(k\wedge\bar{m}) - \mathrm{cc.}\Big)}$$

Area two-form: $\varepsilon = -\mathrm{i} m \wedge \bar{m}$.

Shear and expansion:

$$\ell_A D\ell^A = -\left(\frac{1}{2}\vartheta_{(l)} m + \sigma_{(l)}\bar{m}\right)$$

Gauge symmetries:

- vertical diffeomorphisms $\delta^{diff}_{\xi}[\cdot] = \mathscr{L}_{\xi}[\cdot] : \xi^a \sim l^a \in T\mathscr{N}$
- $U(1)$ transformations $\delta^{U(1)}_{\varphi}[\varkappa_a, m_a] = [-\gamma^{-1}\partial_a\varphi, \mathrm{i}\,\varphi m_a]$
- dilations $\delta^{dilat}_{f}[\varkappa_a, l^a] = [\partial_a f, f l^a]$
- shift symmetry $\delta^{shift}_{\zeta}[\varkappa_a] = \zeta\bar{m}_a + \bar{\zeta} m_a$

Transition to $SL(2,\mathbb{R})$ variables

Covariant vs. kinematical phase space

In gravity, covariant phase-space methods are useful to

- identify gauge symmetries,
- calculate charges,
- derive the first-law of BH thermodynamics.

Less useful to identify Dirac observables and their algebra.

Strategy ahead:

1. embed covariant phase space into larger kinematical phase space.
2. impose constraints that bring us down to physical phase space.

Auxiliary two-dimensional vector space $\mathbb{V}$ with complex basis $(m^i, \bar{m}^i)$, $i = 0, 1$, and internal metric q_{ij}, $q^{ij} : q^{ik} q_{kj} = \delta^i_j$.

Fiducial dyad

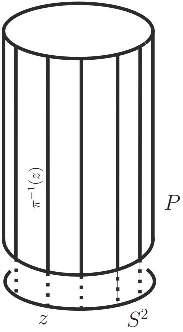
$$e^i_{(o)} = \bar{m}^i \frac{\mathrm{d}z}{1 + |z|^2} + \mathrm{cc.},$$
$$\delta[e^i_{(o)}] = 0.$$

Fiducial area

$$e^i_{(o)} \wedge e^j_{(o)} = \varepsilon^{ij} d^2 v_o.$$

Parametrisation of the dyad

$$\boxed{e^i = \Omega\, S^i{}_j\, e^j_{(o)}.}$$



Basic variables are now: $S^i{}_j : \mathcal{N} \to SL(2, \mathbb{R})$ and conformal factor $\Omega : \mathcal{N} \to \mathbb{R}$.

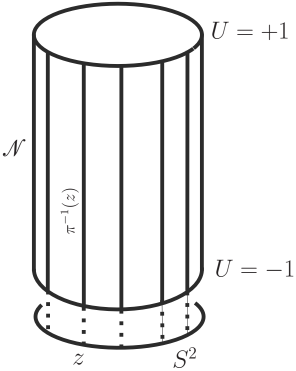
Convenient time variable $U : \mathscr{N} \to \mathbb{R}$, such that

Boundary condition at $\partial\mathscr{N} = \mathscr{C}_+ \cup \mathscr{C}_-$,

$$U(\partial\mathscr{N}, z, \bar{z}) = \pm 1,$$

Non-affinity equals expansion

$$\partial_U^b \nabla_b \partial_U^a = -\frac{1}{2}(\Omega^{-2}\frac{\mathrm{d}}{\mathrm{d}U}\Omega^2)\partial_U^a$$



Nota bene: $\delta U \neq 0$, but $\delta U\big|_{\partial\mathscr{N}} = 0$.

Step 2: Symplectic potential

Quantities with a circumflex are pull-backs to the fibres $\gamma_z = \pi^{-1}(z)$.

$$\boxed{\Theta_{\mathcal{N}} = \frac{1}{8\pi G}\int_{\mathcal{N}} d^2v_o \wedge \left[p_K\, \mathbb{d}\widetilde{K} + \gamma^{-1} E\, \mathbb{d}\widetilde{\Phi} + \widetilde{\Pi}^i{}_j\, \left[S\mathbb{d}S^{-1}\right]^j{}_i\right] + \textit{corner term.}}$$

Abelian variables:

$U(1)$ angle: $\widetilde{\Phi} := -\varphi_{(l)}\varphi^*_{\gamma_z} k$, area: $E := \Omega^2$, lapse: $\widetilde{K} := \text{đ}U \equiv \varphi^*_{\gamma_z} \mathrm{d}U$.

$SL(2,\mathbb{R})$ holonomy flux variables

$$\begin{aligned}
\{\widetilde{\Pi}(x), S(y)\} &= -8\pi G\, X S(y)\, \tilde{\delta}_{\mathcal{N}}(x,y),\\
\{\widetilde{I}(x), S(y)\} &= +4\pi G\, J S(y)\, \tilde{\delta}_{\mathcal{N}}(x,y),\\
\{\widetilde{\Pi}(x), \widetilde{I}(y)\} &= -8\pi \mathrm{i} G\, \widetilde{\Pi}(y)\, \tilde{\delta}_{\mathcal{N}}(x,y),\\
\{\widetilde{\Pi}(x), \widetilde{\bar{\Pi}}(y)\} &= -16\pi \mathrm{i} G\, \widetilde{I}(y)\, \tilde{\delta}_{\mathcal{N}}(x,y),
\end{aligned}$$

Basis in $SL(2,\mathbb{R})$ such that $\widetilde{\Pi}^i{}_j = \widetilde{I} J^i{}_j + \widetilde{\Pi}\bar{X}^i{}_j + \widetilde{\bar{\Pi}} X^i{}_j$,
and $[J, X] = -2\mathrm{i}X$, $[X, \bar{X}] = \mathrm{i}J$.

$U(1)$ Gauss constraint (diagonal part of $SL(2,\mathbb{R})$)

$$\forall\Lambda : G[\Lambda] = \int_{\mathcal{N}} d^2 v_o \wedge \Lambda \left(\widetilde{I} - \frac{1}{2\gamma}\, \text{đ}\, E \right) \stackrel{!}{=} 0,$$

Hamilton constraint/Raychaudhuri equation

$$\forall \xi^a : \pi_* \xi^a = 0 : H_\xi = -\frac{1}{4\pi G} \int_{\mathcal{N}} d^2 v_o \wedge \mathrm{d}U\, \mathscr{L}_\xi[U] \left[\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}U^2} \Omega^2 + \sigma\bar{\sigma} \right] \stackrel{!}{=} 0,$$

Shear in terms of the off-diagonal components of $\mathfrak{sl}(2,\mathbb{R})$-valued momentum

$$\widetilde{\Pi} := \frac{\gamma + \mathrm{i}}{\gamma} \Omega\, \sigma\, \text{đ}U$$

Define $\mathfrak{sl}(2,\mathbb{R})$ connection

$$\text{đ}\, S\cdot S^{-1} =: \widetilde{\varphi} J + \widetilde{h}\bar{X} + \widetilde{\bar{h}} X,$$

Second-class constraints

$$\forall\mu : D[\mu] = \int_{\mathcal{N}} d^2 v_o \wedge \mu\left(\tilde{\Phi} - \tilde{\varphi}\right) \stackrel{!}{=} 0,$$
$$\forall\zeta : V[\bar{\zeta}] = \int_{\mathcal{N}} d^2 v_o \wedge \bar{\zeta}\mathrm{e}^{-2\mathrm{i}\Delta}\left(\Omega^{-1}\widetilde{\Pi} - \frac{\gamma+\mathrm{i}}{\gamma}\Omega\widetilde{h}\right) \stackrel{!}{=} 0,$$
$$\forall\lambda : C[\lambda] = \int_{\mathcal{N}} d^2 v_o \wedge \lambda\left(p_K \widetilde{K} - \text{đ}\, E\right) \stackrel{!}{=} 0,$$

$U(1)$ connection

$$\Delta(u,z,\bar{z}) = \int_{\gamma_z(u)} \widetilde{\varphi},$$

Dirac bracket

Dirac bracket for $SL(2,\mathbb{R})$ holonomy

$$\boxed{\begin{aligned}\left\{S^i{}_m(x), S^j{}_n(y)\right\}^* &= -4\pi G\,\Theta(x,y)\,\delta^{(2)}(x,y)\,\Omega^{-1}(x)\,\Omega^{-1}(y)\\ &\quad\times\left[\mathrm{e}^{-2\,\mathrm{i}\,(\Delta(x)-\Delta(y))}\left[XS(x)\right]^i{}_m\left[\bar{X}S(y)\right]^j{}_n + \mathrm{cc.}\right].\end{aligned}}$$

Dirac observables can be constructed using standard techniques.

Gauge symmetries:

1. $U(1)$ transformations
2. vertical diffeomorphisms along null generators

Summary

Summary

- Action with Barbero–Immirzi parameter γ in causal regions
- γ mixes $U(1)$ frame rotations and dilations. **This is extremely important, it gives a geometric explanation for LQG discreteness of geometry.**
- Kinematical phase space carries $SL(2,\mathbb{R})$ holonomy-flux algebra
- All constraints are polynomials in the fundamental fields.