AIMS-Senegal: Lectures on Special and General Relativity

Outline of the lectures

1. Introduction: The world of Newton
   1.1 Space, Time, inertial frames
   1.2 Energy, momentum
   1.3 Newtonian Gravity

2. Special relativity

2.1 The speed of light is a universal constant

2.2 Spacetime diagrams

2.3 Lorentz transformations and inertial frames of reference

2.4 Addition of velocities

2.5 Time dilatation

2.6 Twin paradox

2.7 Lorentz contraction and invisibility of Lorentz transformations

3. Relativistic particles

3.1 Four-vectors, metric and invariant distance

3.2 Energy-momentum vector, relativistic collisions

3.3 Accelerated Motion and the relativistic rocket equation

4. General relativity

Intro flat space in curved coordinates

4.1 Equivalence principle and metric for accelerated observers

two or three lectures

4.2 Elements of Differential calculus
- Manifolds
- Vectors and Tensors
- Metric
- Covariant derivative
- Torsion
- Curvature

? 4.3 Navier Stokes equation and the energy momentum tensor of fluids

one lecture

4.4 Geodesics and relative acceleration

4.5 Einstein equations

1/2 4.6 Black holes

1/2 4.7 Big bang and cosmology

1. Introduction: the world of Newton

1.1 Space, time, inertial frames

In Newtonian gravity, there is absolute space and time. An event $\mathcal{E}$ is characterised by ~~four numbers~~:

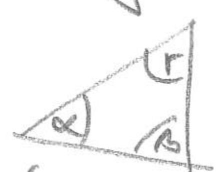
the time $t(\mathcal{E})$ it happens and its position in space

$$\vec{x}(\mathcal{E}) = \begin{pmatrix} x(\mathcal{E}) \\ y(\mathcal{E}) \\ z(\mathcal{E}) \end{pmatrix} = x^i(\mathcal{E}) \quad ; \quad i = 1,2,3$$

Space-time geometry in Newtonian physics

Time is absolute in Newtonian physics: Different observers measure the same time no matter how fast they move.

Newtonian mechanics assumes that the geometry of $\vec{x}$-space is Euclidean:

- parallel lines never intersect 
- angles in a triangle sum up to $\pi = 180°$:  $\alpha + \beta + \gamma = \pi$
- a circle of radius r has circumference $2\pi r = C(r)$
- distances are measured by the formula

$$d(\vec{x}, \vec{y}) = \sqrt{\delta_{ij}(x^i - y^i)(x^j - y^j)} = |\vec{x} - \vec{y}|$$

- angles are measured by the formula

$$|\vec{x}| \cdot |\vec{y}| \cos(\varphi(\vec{x}, \vec{y})) = \delta_{ij} x^i y^j = \vec{x} \cdot \vec{y}$$

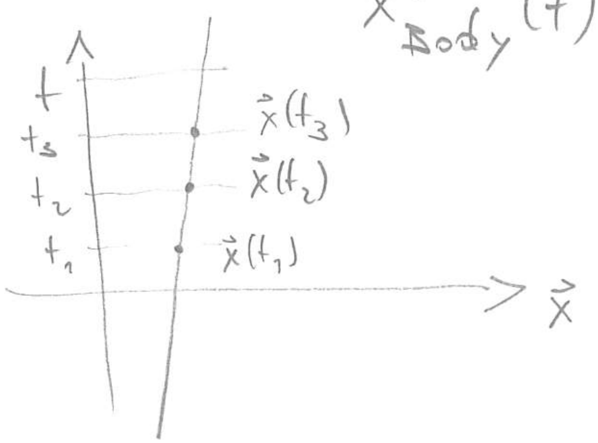
Inertial observers, Newton's first law

Newton's first law: a body either remains at rest or moves at constant velocity, unless acted upon by force.

Thanks to the existence of absolute time and space in Newtonian's gravity, we can translate Newton's first law into a mathematical statement

$$\frac{d^2}{dt^2} X^i_{Body}(t) = 0$$

$$X^i_{Body}(t) = X^i_0 + t \cdot v^i$$



time and distances measured

Inertial frame: by an observer that

either remains at rest or moves at constant velocity in space define an inertial frame

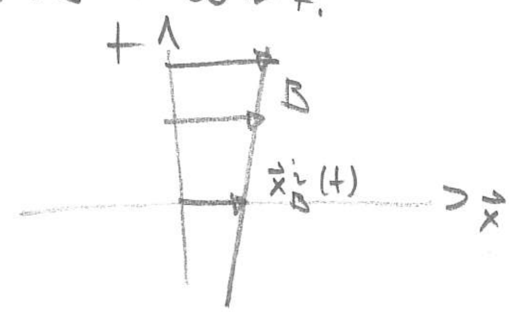
observer A

$$\begin{pmatrix} t \\ x^i \end{pmatrix}$$

observer B

$$\begin{pmatrix} \tilde{t} \\ \tilde{x}^i \end{pmatrix}$$

coordinates of observer Bob w.r.t. observer Alice:



$$x_B^i(t) = x_0^i + v_B^i t$$

coordinate transformation from B to A

$$\tilde{t} = t - t_0$$ (B started his clock t_0 seconds after A)

$$\tilde{x}^i = x^i - x_0^i - v_B^i t$$

Any two inertial frames are related by a combination of any of the following 4 transformations

1. translation of time (resetting of clocks)

$$\tilde{t} = t - t_0$$

2. translation in space (shifting the origin of the coordinate system)

$$\tilde{x}^i = x^i - x_0$$

3. Galilean boost (transformation into frames that move at constant velocity or are at rest)

$$\tilde{x}^i = x^i - v^i t$$

4. Rotations

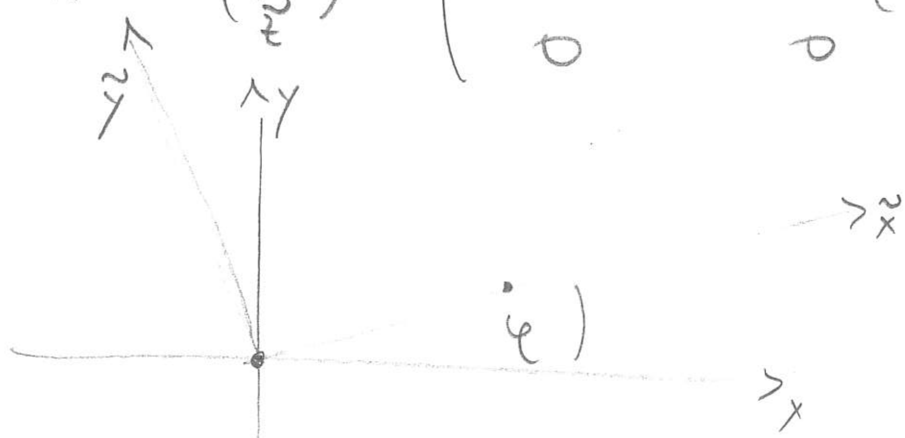
$$\tilde{x}^i = R^i{}_j\, x^j = \sum_{j=1}^{3} R^i{}_j\, x^j$$

where $R^i{}_j$ is a rotation matrix

row column

Example (rotation around z-axis)

$$\tilde{x}^i = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \cos\varphi & +\sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



1.2 Energy and momentum

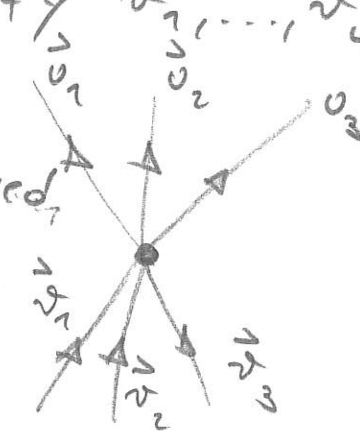
The kinetic energy of a particle of mass m and velocity $\vec{v} = \frac{d}{dt}\vec{x}$ with respect to an inertial system with coordinates $\left\{\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\}$ is given by

$$E_{kin} = \frac{1}{2} m \vec{v}^2$$

The momentum of a particle of mass m and velocity $\vec{v} = \frac{d}{dt}\vec{x}$ with respect to an inertial system $\left\{\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\}$ is given by

$$\vec{p} = m\vec{v}$$

If N particles with mass $m_1, \dots, m_N$ and velocity $\vec{v}_1, \dots, \vec{v}_N$ collide, the total energy and momentum are conserved,



$$E_{in} = \frac{1}{2} \sum_{i=1}^{N} m_i \vec{v}_i^{\,2}$$

$$\vec{p}_{in} = \sum_{i=1}^{N} m_i \vec{v}_i$$

$$E_{out} = \frac{1}{2} \sum_{i=1}^{N} m_i \vec{v}_i^{\,2}$$

$$\vec{p}_{out} = \sum_{i=1}^{N} m_i \vec{v}_i$$

$$\boxed{\begin{aligned} E_{out} &= E_{in} \\ \vec{p}_{out} &= \vec{p}_{in} \end{aligned}}$$ (Newton's third law)

where $(\vec{v}_1, \vec{v}_2, \dots)$ are the velocities at late times

Example: Rocket equation

$$\Delta p = \Delta p_{ex}$$

$$\frac{\Delta p}{\Delta t} = \frac{\Delta p_{ex}}{\Delta t} = \frac{\Delta m}{\Delta t}(v - v_{ex})$$

$$\frac{dp}{dt} = \frac{dm}{dt}(v - v_{ex})$$

v_{ex} .... exhaust velocity
m ..... total rocket mass
m_0 .... ~~dry~~ payload + mass of the rocket
m_{fuel} ... mass of rocket fuel

$p = mv$ .... momentum of the rocket
p_{ex} ..... momentum of exhaust

we thus have

$$\frac{d}{dt}p = \frac{d}{dt}(mv) = \frac{dm}{dt}(v - v_{ex})$$

$$\frac{d}{dt}v = -v_{ex}\frac{1}{m}\frac{dm}{dt}$$

$$dv = -v_{ex}\frac{1}{m}dm$$

$$\int_0^{v_{final}} dv = -v_{ex}\int_{m_0+m_{fuel}}^{m_0}\frac{dm}{m}$$

$$v_{final} = -v_{ex}\ln\left(\frac{m_0}{m_0+m_{fuel}}\right) =$$

$$= v_{ex}\ln\left(1+\frac{m_{fuel}}{m_0}\right).$$

$$\boxed{v_{final} = v_{ex}\ln\left(1+\frac{m_{fuel}}{m_0}\right)}$$

1.3 Newtonian Gravity

Newton's second law: the product of the mass of a particle times the its acceleration $\vec{a}$ with respect to an inertial frame of reference is proportional to the sum of forces acted upon the particle

$$\vec{F} = m\vec{a} = m\frac{d^2}{dt^2}\vec{x}$$

Gravitational force $\vec{F}_{grav}$

always attractive

The strength of the gravitational force between two particles of mass m_1 and m_2 is inversely proportional to the square distance r^2 between the two particles

$$F_{grav} = \frac{G m_1 m_2}{r^2}$$

If $M = m_1$ lies in the origin a particle with mass m experiences the force

$$F^i = + \frac{GMm}{r^2} e^i = - \frac{GMm}{r^3} x^i$$

$$\boxed{G \approx 6{,}674 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}}$$

circular orbits first

Gravitational potential and conservation of energy

<u>work</u> $W = \text{Force} \times \text{displacement}$

$$W = \vec{F} \cdot \Delta\vec{x}$$

$$W = \int_{\gamma(x_0 \to x_1)} dt\, F_i \frac{dx^i}{dt} = m \int_{\gamma(x_0 \to x_1)} dt\, \frac{d^2x^i}{dt^2} \frac{dx_i}{dt} =$$

$$= \frac{m}{2} \int_{\gamma(x_0 \to x_1)} dt\, \frac{d}{dt}\left(\frac{dx^i}{dt} \frac{dx_i}{dt} \right) = E_{kin}\big|_{x_1} - E_{kin}\big|_{x_0} =$$

$$= -GMm \int_{\gamma(x_0 \to x_1)} dt\, \frac{1}{|\vec{x}|^3} x_i \frac{dx^i}{dt} =$$

$$= +GMm \int_{\gamma(x_0 \to x_1)} dt\, \frac{d}{dt} \frac{1}{|\vec{x}|} =$$

$$= GMm \left(\frac{1}{|\vec{x}_1|} - \frac{1}{|\vec{x}_0|} \right)$$

total

Energy

$$E = E_{kin} + E_{pot} = \text{const.}$$

$$E_{kin} = \frac{1}{2} m \vec{v}^2$$

$$E_{pot} = -\frac{GMm}{|\vec{x}|} = m\,\Phi(\vec{x})$$

(mass M, radius R)

Escape velocity

of a planet

To leave the gravitational pull a body must reach its escape velocity, which can be calculated by demanding that the object comes at rest at infinite distance from the planet

$$E_\infty = \frac{1}{2} m \vec{v}_\infty^2 - \frac{GMm}{|\vec{x}|_\infty} = 0$$

$$= \frac{1}{2} m v^2(R) - \frac{GMm}{R} = 0$$

$$\boxed{v_{escape}(R) = \sqrt{\frac{2GM}{R}}}$$

Circular
~~Radial~~ orbits

$$F^i = m a^i = m \frac{d^2 x^i}{dt^2} =$$

$$= -\frac{GMm}{|\vec{x}|^3} x^i(t) ; \quad \boxed{\frac{d^2 x^i}{dt^2} = -\frac{GM}{|\vec{x}(t)|^3} x^i(t)}$$

Ansatz

$$\vec{x}(t) = x^i(t) = \begin{pmatrix} x(t) \\ y(t) \\ 0 \end{pmatrix} = r \begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix}$$

$$\vec{v}(t) = \frac{d}{dt}\vec{x}(t) = r\omega \begin{pmatrix} -\sin\omega t \\ \cos\omega t \\ 0 \end{pmatrix}$$

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = -r\omega^2 \begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix}$$

$$r\omega^2 = +\frac{GM}{r^2}$$

$$\boxed{\omega^2 = \frac{GM}{r^3}}$$

Orbital period of the ISS

$$\omega = \frac{2\pi}{T}$$

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r^3}{GM}}} \approx 1{,}5\ h$$

Velocity to reach ISS

$$v = r\omega = \sqrt{\frac{GM}{r}} = v_{equator} + v_{rocket}$$

$$v_{rocket} = \sqrt{\frac{GM}{R+h}} - \frac{2\pi R}{24\cdot[h]} \approx 7688\ \frac{m}{s}$$

$R \approx 6370$ km
$h \approx 400$ km
$M \approx 6\cdot 10^{24}$ kg

Payload for rocket (Falcon 9 Block 5) to ISS

$$v_{final} = v_{ex} \ln\left(1 + \frac{m_{fuel}}{m_0}\right)$$

$$m_0 = m_{rocket} + m_{payload}$$

$$m_{payload} = \frac{m_{fuel}}{e^{v_{final}/v_{ex}} - 1} - m_{rocket}$$

$$v_{final} = 7688 \ \frac{m}{s}$$

$$v_{ex} = 3000 \ \frac{m}{s}$$

$$m_{fuel} = 410\,900 \ kg$$

$$m_{rocket} = 22\,200 \ kg$$

$$\boxed{m_{payload} \approx 12000 \ kg}$$ ← Payload to ISS

Newtonian field equations

gravitational potential for a single point mass at the origin

$$\phi(\vec{x}) = -\frac{GM}{r}$$

Dirac-Delta

$$\int d^3x\, f(\vec{x}) \Delta \frac{1}{|\vec{x}|} = -\int d^3x\, (\vec{\nabla} f)(\vec{x}) \vec{\nabla} \frac{1}{|\vec{x}|} =$$

$$= \int d^3x\, (\vec{\nabla} f)(\vec{x}) \frac{\vec{x}}{|\vec{x}|^3} = \int d^2\Omega \int_0^{+\infty} dr\, \frac{d}{dr} f =$$

$$= -4\pi f(\vec{0})$$

Laplace Equation for the gravitational potential

$$\boxed{\Delta \phi = +4\pi G \rho}$$

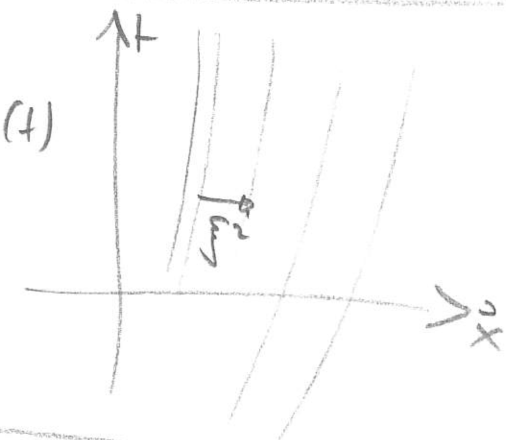
Equation for particles in the gravitational field

$$\boxed{m \frac{d^2}{dt^2} x^i = - m\, \partial^i \phi}$$

Relative acceleration – tidal forces

$x^i_\varepsilon(t)\ ; \quad x^i_{\varepsilon=0}(t) = x^i(t)$

$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} x^i_\varepsilon(t) = \xi^i(t)$



$$\boxed{\frac{d^2}{dt^2} \xi^i(t) = -\left(\partial^i \partial_j \phi\right)(\vec{x}(t))\, \xi^j(t)}$$