Simplicial Graviton from Selfdual Ashtekar Variables

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Outline and Motivation

Outline

Quantum gravity in finite regions — a paradigm of *quasi-local holography*.

- It is *quasi-local* rather than *local*, because observables are attached to finite regions rather than points on the manifold.
- It is *holographic*, because evolution is studied through the exchange of charges at the boundary seperating system and environment.

Paradigm connecting research on quantum reference frames, quantum gravity, observables, holography.

In this talk, I will focus on only one recent development: Quasi-local regularization of constraint algebra for selfdual gravity with potential connection to earlier results by A. Ashtekar and M. Varadarajan. Simplicial version of time evolution as spatial diffeomorphism.

*Laurent Freidel, Marc Geiller, ww, Corner symmetry and quantum geometry, Springer Handbook od Spacetime (2023), arXiv:2302.12799.
*ww, Simplicial Graviton from Selfdual Ashtekar Variables (2023), arXiv:2305.01803.
*Abhay Ashtekar, Madhavan Varadarajan, Gravitational dynamics: A novel shift in the Hamiltonian paradigm, Universe 7, 13 (2021), arXiv:2012.12094.
*Valentin Bonzom, Laurent Freidel, The Hamiltonian constraint in 3d Riemannian loop quantum gravity, Class. Quant. Grav. 28 (2011), arXiv:arXiv:1101.3524.

Back to selfdual variables

[Ashtekar (1986)]: general relativity put on the phase space of an $SL(2,\mathbb{C})$ Yang–Mills theory.

$$\left\{\tilde{E}_i{}^a(x), A^j{}_b(y)\right\} = 8\pi \mathrm{i} G \delta_i^j \delta_b^a \tilde{\delta}^{(3)}(x,y),$$

$a, b, c, \ldots$ are tangent indices and $i, j, \ldots$ refer to the internal $SL(2,\mathbb{C})$ directions. There are three of them as $SL(2,\mathbb{C})$ is treated here as complexifization of $SU(2)$ (complex structure $\leftrightsquigarrow$ normal vector to Σ).

The constraints are the simplest possible gauge-invariant polynomials on the now complexified phase space.

$$\begin{aligned} \text{Gauss constraint:} \quad & \tilde{\mathscr{G}}_i = D_a \tilde{E}_i{}^a = 0, \\ \text{Vector constraint:} \quad & \tilde{H}_a = F^i{}_{ab}\, \tilde{E}_i{}^b = 0, \\ \text{Hamiltonian constraint:} \quad & \tilde{\tilde{H}} = \epsilon_i{}^{lm} F^i{}_{ab}\, \tilde{E}_l{}^a \tilde{E}_m{}^b. \end{aligned}$$

ADM-type of action, metric on superspace

N.B.: Scalar constraint defines an inverse super-metric $G = \frac{1}{2}\epsilon_i{}^{lm} F^i{}_{ab}[A] \frac{\delta}{\delta A^i{}_a} \otimes \frac{\delta}{\delta A^j{}_b}$ on configuration space. ADM-type action

$$S = \int \mathrm{d}t \int_\Sigma \Big(\tilde{E}_i{}^a \underbrace{\left(\tfrac{\mathrm{d}}{\mathrm{d}t} A^i{}_a - D_a \Lambda^i + F^i{}_{ab} N^b\right)}_{\frac{D}{\mathrm{d}t} A^i{}_a} - \underset{\sim}{N} \epsilon_i{}^{lm} F^i{}_{ab}[A] \tilde{E}_l{}^a \tilde{E}_m{}^b \Big).$$

Compare this with the worldline action for $I = 1, \ldots, K$ (uncoupled) massless particles

$$S = \int \mathrm{d}t \Big(\sum_{I=1}^{K} p^I_\mu(t) \frac{\mathrm{d}}{\mathrm{d}t} q^\mu_I(t) - \sum_{I=1}^{K} N_I \, g^{\mu\nu}(q_I(t)) p^I_\mu(t) p^I_\nu(t) \Big).$$

Physical spacetimes carve out null geodesic in the space of self-dual connections.

Constraint algebra

All constraints are first-class (as in Yang–Mills). However, also very different from Yang–Mills, as there are structure functions rather than structure constants.

$$\{G_i[\Lambda^i], G_j[\mathrm{M}^j]\} = -8\pi \mathrm{i} G\, G_i\big[[\Lambda, \mathrm{M}]^i\big],$$
$$\{H_a[N^a], H_b[M^b]\} = -8\pi \mathrm{i} G\Big(H_a\big[[N, M]^a\big] - G_i\big[F^i{}_{ab}\, N^a M^b\big]\Big),$$
$$\{H_a[N^a], H[\underset{\sim}{N}]\} = -8\pi \mathrm{i} G\Big(H[\mathscr{L}_{\vec{N}} \underset{\sim}{N}] + G_i\big[\epsilon_j{}^{ki} F^j{}_{ab}\, \tilde{E}_k{}^a \underset{\sim}{N} N^b\big]\Big),$$
$$\{H[\underset{\sim}{N}], H[\underset{\sim}{M}]\} = +8\pi \mathrm{i} G\, H_a\big[[\underset{\sim}{N}, \underset{\sim}{M}]^a\big],$$

where e.g. $G_i[\Lambda^i] = \int_\Sigma \tilde{\mathscr{G}}_i \Lambda^i$, and $[\underset{\sim}{N}, \underset{\sim}{M}]^a = \delta^{ij}\, \tilde{E}_i{}^a \tilde{E}_j{}^b (\underset{\sim}{N} D_b \underset{\sim}{M} - \underset{\sim}{M} D_b \underset{\sim}{N})$.

N.B.: Algebra still regular for degenerate geometries (squashed triads) $\frac{1}{3!}\epsilon^{ijk} \underset{\sim}{\varepsilon}_{abc} \tilde{E}_i{}^a \tilde{E}_j{}^b \tilde{E}_k{}^c = 0$.

Physical modes per point of Σ

Counting *complex* degrees of freedom:

Each point x on Σ carries kinematical variables $A^i{}_a(x)$ and $\tilde{E}_i{}^a(x)$.

Kinematical degrees of freedom per points of Σ:

$$3 \times 3 = 9 \quad \text{(complex degrees of freedom)}$$

Physical degrees of freedom per points of Σ:

$$9 - 3 - 3 - 1 = 2 \quad \text{(complex degrees of freedom)}$$

These are the two modes of polarization of the non-linear graviton.

We have no complete set of Dirac variables that could access these modes at the full non-linear level. Standard constructions of Dirac observables and physical phase space rely on additional auxiliary structures, e.g. boundary conditions, asymptotic falloff conditions, perturbation theory.

N.B.: Phase space is complex, all constraints are analytic functionals on phase space. Going back to metric GR requires additional reality conditions.

Main strategy ahead

All physics relies on truncations. Idea: find a way to isolate the two physical modes at the discretised level.

Basic strategy: introduce simplicial decomposition. Blow up points on the initial manifold and replace them by simplicial building blocks (tetrahedra).

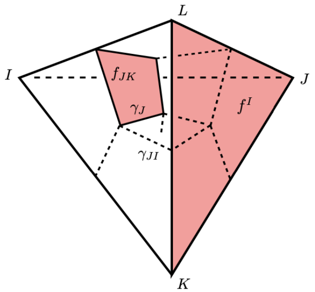
Perhaps overly naïve, but if it works we could expect: each simplicial cell represents a fundamental *atom of geometry*. Ignoring for a moment additional boundary modes perhaps necessary, each such *atom of space* is expected to carry two physical modes (four complex phase space dimensions). Thus localizing radiative modes in each simplicial cell.

Step 1: new regularization of the constraints

Lattice and dual lattice in a single cell

Introduce a simplicial discretisation of Σ. Consider a single building block, an elementary tetrahedron $T \subset \Sigma$.

- Boundary of the tetrahedron consists of four triangles f^1, f^2, f^3, f^4.
- Each such *face* f^I is dual to a *half link* γ_I connecting the centroid of T with the centroid of f^I.
- *Boundary links* $\gamma_{JI} \subset \partial T$ connect the centroid of f^I with the centroid of f^J along the boundary of T.
- The dual faces f_{JK} are bounded by the *loop* $\gamma_J^{-1} \circ \gamma_{KJ} \circ \gamma_K$.



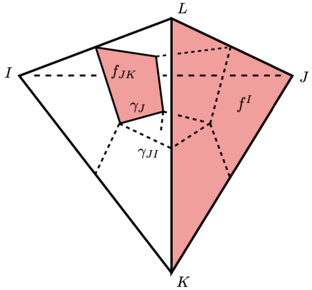
Comments:

- We need to speak about such internal loops (wedge holonomies)—otherwise it is impossible to regularise the field strength ${F^i}_{ab}$ in a given cell T.
- Without wedge holonomies, non-local construction necessary in which the field strength is smeared over a plaquette connecting many tetrahedra.
- We want to avoid such non-local construction, otherwise seems hopeless to do constraint analysis on arbitrary triangulation.

This is a departure from *Regge calculus*, where each building block is assumed to be flat. No such assumption here. Hence, there is curvature in each bulding block.

- Bulk holonomies: $h_I = \mathrm{Pexp}\big(-\int_{\gamma_I} A\big)$.
- Boundary holonomies:
  $h_{IJ} = \mathrm{Pexp}\big(-\int_{\gamma_{IJ}} A\big) = g_J^{-1} g_I$.
- *Assumption:* All curvature concentrated in the bulk. Boundary flatness:
  $h_{IK} h_{KJ} h_{JI} = \mathbb{1}$.
- Bulk curvature:

$$F_{IJ} := \mathrm{Pexp}\Big(-\oint_{f_{IJ}} A\Big) = h_J^{-1} h_{IJ} h_I .$$

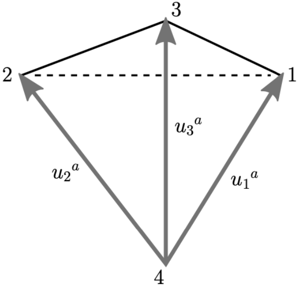


- Non-abelian Stokes's theorem:

$$\mathrm{Pexp}\Big(-\oint_{\partial f} A\Big) = \mathrm{Sexp}\Big(-\int_f F\Big) \approx \mathbb{1} - \int_f F^i \tau_i + \dots$$

Besides the holonomies (magnetic fluxes), we have the electric fluxes.

- Ashtekar electric flux: $E_i{}^I = \int_{f^I} (E_j)_x [g^{-1}(x) g_I h_I]^j{}_i$.
- Adapted triad: $u_1{}^a = X_1^a - X_4^a, \quad u_2{}^a = X_2^a - X_4^a, \quad u_3{}^a = X_3^a - X_4^a$.
- Dual triad: $u^\mu{}_a$: $u_\alpha{}^a u^\alpha{}_b = \delta^a_b$.
- Adapted coordinates (u^1, u^2, u^3) such that tetrahedron is the point set $u^\alpha > 0$, $\sum_{\alpha=1}^3 u^\alpha < 1$.



- Electric and magnetic fluxes:

$$E_i{}^1 \approx -\frac{1}{2} \underset{\sim}{\epsilon}_{abc} u_2{}^a u_3{}^b u_\mu{}^c u^\mu{}_a \tilde{E}_i{}^a \Big|_c = -\frac{1}{2} (d^3 u)^{-1} u^1{}_a \tilde{E}_i{}^a \Big|_c,$$

$$\frac{1}{6} F^i{}_{ab}\, u_\alpha{}^a u_\beta{}^b \Big|_c \approx F^i[f_{\alpha\beta}] - F^i[f_{4\beta}] - F^i[f_{\alpha 4}], \text{ where: } F^i[f] = \int_f F^i.$$

Regularisation of the Gauss constraint

In the continuum, the smeared Gauss constraint is

$$G_i[\Lambda^i] = \int_\Sigma \Lambda^i D_a \tilde{E}_i{}^a = 0.$$

Choose a test function Λ^i that vanishes everywhere except in T. At the discretised level, the constraint is then well approximated by the closure constraint in each tetrahedron

$$\boxed{G_i^T[\Lambda^i] = \sum_{I=1}^{4} \Lambda^i E_i{}^I = 0.}$$

N.B.: If we also impose reality conditions on $E_i{}^I$, the closure constraint allows to assign a geometric data (edge lengths) to each tetrahedron via Minkowski theorem.

$$\text{if} \quad G_i^T = 0 \quad \text{and} \quad E_i{}^I = \bar{E}_i{}^I : E_i{}^I = n_i{}^I(\{\ell_e\}, R)\,\text{area}(\{\ell_e\}).$$

In the continuum, the smeared vector constraint is

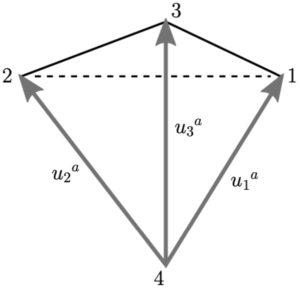
$$H_a[N^a] = \int_\Sigma N^a F^i_{\ ab}\, \tilde{E}_i^{\ b}.$$

- Set $N^a = 0$ expect in T.
- Set in T: $N^a = N^\mu u_\mu^{\ a}$, $\mu = 1, 2, 3$.
- Use non-abelian Stokes's theorem

$$F_{IJ} = \mathrm{Pexp}\Big(-\oint_{\partial f_{IJ}} A \Big) \approx \mathbb{1} - \oint_{f_{IJ}} F.$$

- Introduce fourth auxiliary direction

$$N^4 = -\sum_{\mu=1}^{3} N^\mu.$$



- Regularized vector constraint

$$\boxed{\int_T N^a F^i_{\ ab}\, E_i^{\ b} \approx -4 \sum_{I,J=1}^{4} N^I \mathrm{Tr}\big(\tau^j F_{IJ}\big) E_j^{\ J}.}$$

The scalar constraint is smeared against an *inverse density*

$$H[\underset{\sim}{N}] = \int_\Sigma \underset{\sim}{N} \epsilon_i{}^{lm} F^i{}_{ab} \tilde{E}_l{}^a \tilde{E}_m{}^b .$$

We split this integral into two parts. We assume $\underset{\sim}{N} = 0$ outside T. Within T, we regularize it by introducing the smeared quantity

$$N_T := \Big[\int_T \underset{\sim}{N}^{-1} \Big]^{-1},$$

which is independnet of metric and connection (a c number).

Assuming the fields are slowly varying in T, we obtain the regularized constraint

$$\boxed{\int_T \underset{\sim}{N} \epsilon_i{}^{lm} F^i{}_{ab} \tilde{E}_l{}^a \tilde{E}_m{}^b \approx \frac{8}{3} \sum_{I,J=1}^{4} N_T \mathrm{Tr}\big(\tau^j F_{IJ} \tau^i\big) E_i{}^I E_j{}^J .}$$

Step 2: additional closure constraint

Is there the right number of physical modes per lattice site?

The commutation relations for the *half holonomies* and *fluxes* define the phase space $T^*SL(2,\mathbb{C})^4$.

Is there a chance that we obtain the correct number of physical modes per lattice site?

The contribution to the symplectic potential from each lattice site is

$$\Theta_T(\delta) = 16\pi\mathrm{i}G\sum_{I=1}^{4} E_i{}^I \mathrm{Tr}\big(\tau^i h_I^{-1}\delta h_I\big).$$

Now all four directions are treated as functionally independent. Yet in the discrete, the tangent indices $a, b, c, \dots$ refer to a three-dimensional space.

Tension: In the continuum, we have

$$\Theta_M(\delta) = 8\pi\mathrm{i}G\int_M \tilde{E}_i{}^a \delta A^i{}_a.$$

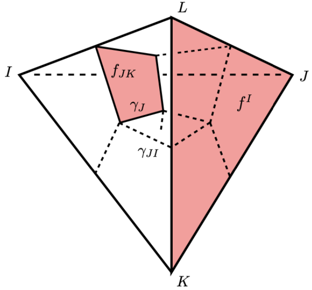
Assuming all discretised constraints are first-class, we would be left with three additional spurious degrees of freedom:
$3 \times 4 - 3 - 3 - 1 = 5 = 2 + 3$.

To remove the additional unphysical modes, it seems necessary to add one additional closure constraint.

Consider the dressed closure constraint (sort-of Bianchi identity?)

$$\sum_{K=1}^{4} G_{k(K)} := \frac{1}{4} \sum_{I,K=1}^{4} [F_{KI}]^{i}{}_{k} E_{i}{}^{I}.$$

- In the continuum limit, this constraint is functionally dependent of the other constraints.
- It becomes proportional to the usual closure constraint.
- $[F_{KI}]^{i}{}_{k}$ is the adjoint representation: $h^{-1}\tau^{i}h = [h]^{i}{}_{j}\tau^{j}$ with τ_i Pauli matrices.
- Furthermore, for Regge-like curvature, the dressed closure constraint is again proportional to the usual closure constraint:



Regge-like configurations: $\quad E_{i}{}^{I} = [F_{KI}]^{i}{}_{k} E_{i}{}^{I}.$

All constraints first class

The set of constraints per each tetrahedron is first-class. With constraints:

closure constraint: $G_i = \sum_{I=1}^{4} E_i{}^I = 0,$

dressed closure: $G_{i(K)} = \sum_{I=1}^{4} [F_{KI}]^i{}_k E_i{}^I = 0,$

vector constraint: $H_I[N^I] = -4 \sum_{I,J=1}^{4} N^I \mathrm{Tr}\big(F_{IJ}\tau^j\big) E_j{}^J = 0, \ \forall N^I : \sum_{I=1}^{4} N^I = 0,$

scalar constraint: $H = \frac{8}{3} \sum_{I,J=1}^{4} \mathrm{Tr}\big(\tau^i F_{JI} \tau^j\big) E_i{}^I E_j{}^J = 0.$

For example:

$$\big\{H_I[N^I], H_J[M^J]\big\} = -8\pi \mathrm{i} G H_I[N,M]^I + \text{closure constraints},$$

$$[N,M]^I = \sum_{J=1}^{4} \Big(N^J \mathrm{Tr}(F_{JI}) M^I - M^J \mathrm{Tr}(F_{JI}) N^I\Big).$$

Result relies on two additional constraints

By adding two additional conditions, we obtain a closed algebra:

- dressed closure constraint (a central term): $\sum_{I=1}^{4}[F_{KI}]^{i}{}_{k}E_{i}{}^{I}=0$.
- Boundary flatness: $\varphi_T^{*}A=g^{-1}\mathrm{d}g$.

The reduced phase space has 2×2 complex dimensions, i.e. the *simplicial graviton for selfdual gravity.*

What about gravity in $2 + 1$ dimensions?

Test: Repeating the construction for 3d gravity

Three-dimensional (Euclidean) gravity admits formulation in terms of Ashtekar's connection dynamics [A. Ashtekar, R. Loll (1994)]:

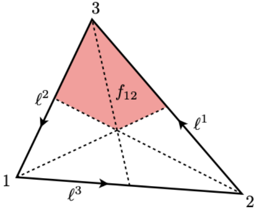
- Kinematical phase space of $SU(2)$ gauge connection and electric field: $\{\tilde{E}_i{}^a(p), A^j{}_b(q)\} = 8\pi G\, \tilde{\delta}^{(2)}(p,q)$,
- Constraints just the same as in four-dimensional selfdual theory:
  - Gauss: $D_a \tilde{E}_i{}^a = 0$,
  - Vector: $F^i{}_{ab}\, \tilde{E}_i{}^b = 0$,
  - Hamilton: $\epsilon_i{}^{jk} F^i{}_{ab}\, \tilde{E}_j{}^a \tilde{E}_k{}^b = 0$.
- No local degrees of freedom: $3 \times 2 - 3 - 2 - 1 = 0$.

Hamiltonian lattice approach. Introduce triangulation of initial surface M.

Each triangle equipped with phase space $T^* SU(2)^3$.

As before, we split each simplicial building block, i.e. every triangle $\triangle$, into smaller wedges f_{IJ}.

- Wedge holonomies: $F_{IJ} = \mathrm{Pexp}(-\oint_{f_{IJ}} A)$.
- Boundary flatness $\mathrm{Pexp}(-\oint_{\partial\Delta} A) = \mathbb{1}$ does not imply wedge flatness.
- Constraints assume same form as before, but now straight-forward to solve.
- Constraints impose wedge flatness $F_{IJ} = \mathbb{1}$.



In three dimensions:

- Closure: $\sum_{I=1}^{3} E_i{}^I = 0$.
- Dressed closure: $\sum_{I=1}^{3}[F_{KI}]^j{}_k E_j{}^I = 0$.
- Vector: $\sum_{I,J=1}^{3} \mathrm{Tr}\big(\tau^i F_{IJ}\big) N^I E_i{}^J = 0, \quad \forall N^I : \sum_{I=1}^{3} N^I = 0$.
- Hamilton: $\sum_{I,J=1}^{3} \mathrm{Tr}\left(\tau^j F_{IJ} \tau^i\right) E_i{}^I E_j{}^J = 0$

N.B. Dressed closure implies $F_{IJ} = \exp(-\mu^J E_j{}^J \tau^j) \exp(\mu^I E_i{}^I \tau^i)$. Scalar and vector constraint imply, in turn, $\sin(\frac{\mu^I}{2}) = 0$, i.e. flatness of wedge holonomies.

The (unique) quantum state $\Omega_\triangle$ for a single triangle $\triangle$ that satisfies $\widehat{F_{IJ}}|\Omega_\triangle\rangle = |\Omega_\triangle\rangle$ defines the BF vacuum:

$$\langle h_1, h_2, h_3; g_1, g_2, g_3|\Omega_\triangle\rangle = \delta_{SU(2)}(F_{12})\,\delta_{SU(2)}(F_{23})\,\delta_{SU(2)}(F_{31}),\ F_{IJ} = g_J^{-1}h_J^{-1}h_I g_I$$

The state for an entire triangulation is built by taking the tensor product over all triangles and tracing over boundary modes

$$|\Omega\rangle = \prod_{e:edges}\int_{SU(2)} dg_{s(e)}\int_{SU(2)} dg_{t(e)}\,\delta_{SU(2)}(g_{s(e)}^{-1}g_{t(e)})\,\langle\{g_e\}|\Omega_{\triangle_1},\Omega_{\triangle_2},\ldots\rangle.$$

Conjecture: Same construction possible in $3+1$, but now there are infinitely many allowed physical states $|\Omega_T^\sigma\rangle$ labelled by radiative data σ for each tetrahedron. Superpositions of spin networks, *"warp-network-states"* ...

$$|\sigma_1,\sigma_2,\ldots\rangle = \prod_{e:edges}\int_{SL(2,\mathbb{C})} dg_{s(e)}\int_{SL(2,\mathbb{C})} dg_{t(e)}\,\delta_{SL(2,\mathbb{C})}(g_{s(e)}^{-1}g_{t(e)})\,\langle\{g_e\}|\Omega_{T_1}^{\sigma_1},\Omega_{T_2}^{\sigma_2},\ldots\rangle$$

Main task ahead

The main open problem is how to glue adjacent tetrahedra.

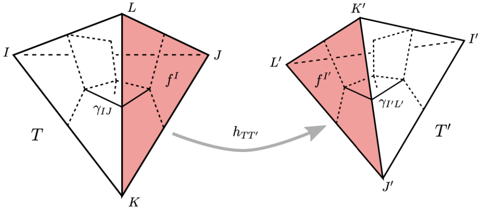
Necessary to go beyond Hamiltonian analysis for a single building block.

Action for a single tetrahedron

$$S[\underline{E}, \underline{h}, \underline{N}, \underline{g}] = \int_{\mathbb{R}} \mathrm{d}t \left(\Theta_{\underline{E},\underline{h}} \left(\frac{\mathrm{d}}{\mathrm{d}t} \right) - C_A(\underline{E}, \underline{h}, \underline{g}) N^A \right).$$

Coupled action from gluing tetrahedra together

$$S_\Delta[\underline{E}, \underline{h}, \underline{N}, \underline{\lambda}] = \sum_{T \in \Delta_3} S[\underline{E}_T, \underline{h}_T, \underline{N}_T, \underline{g}_T] - \sum_{e \in \Delta_1^*} \lambda_e^i \mathrm{Tr}(\tau_i g_{s(e)} g_{t(e)}^{-1}).$$



Summary:

1. New (quasi-local) regularization of the constraints. Possible connections to tensor networks, spinfoams, group field theory, quantum cosmology.
2. Regularisation possible only by introducing additional boundary modes (here: edge modes $g_e \in SL(2, \mathbb{C})$).
3. Additional closure constraint necessary:
   - Otherwise algebra does not close
   - Otherwise counting does not match two physical modes of the continuum
4. In three spacetime dimensions, construction agrees with known results [B. Dittrich, M. Geiller, *BF vacuum* (2014)].

Main open problems:

1. Connection to real variables. Reality conditions. Barbero–Immirzi parameter. Strategy:
   - Momentum shifted: $\tilde{E}_i{}^a \to \frac{\beta+\mathrm{i}}{\mathrm{i}\beta}\tilde{\Pi}_i{}^a$, $\{\Pi, A\} = 1 = \{\bar{\Pi}, \bar{A}\}$.
   - Reality conditions: $\frac{\beta}{\beta+\mathrm{i}}\tilde{\Pi}_i{}^a + \mathrm{cc.} = 0$.
   - Constraints: $H_\beta = \frac{\beta}{\beta+\mathrm{i}}H_{\mathbb{C}} + \mathrm{cc.} = 0$.
2. Gluing of adjacent tetrahedra.
3. Matter couplings.
4. Connection to GFTs, spinfoams.
5. ...