Loop quantum gravity and the continuum

Wolfgang Wieland
Perimeter Institute , Waterloo (ON, Canada)

TU Wien (Austria, EU)

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A bit of background...

- Canonical (Ashtekar) variables: $A^i{}_a = \Gamma^i{}_a[E] + \mathrm{i}K^i{}_a$, $E_i{}^a = d^3v\, e_i{}^a$
- Poisson brackets as in Yang — Mills:
  $\{E_i{}^a(x), A^j{}_b(y)\} = 8\pi \mathrm{i} G \delta^i_j \delta^a_b \delta(x,y)$
- But also very different from Yang – Mills: the Hamiltonian is a sum of constraints (+boundary term at infinity),

$$G_i = D_a E_i{}^a = 0 \quad \textit{(generators of } SL(2,\mathbb{C}) \textit{ gauge transformations)}$$

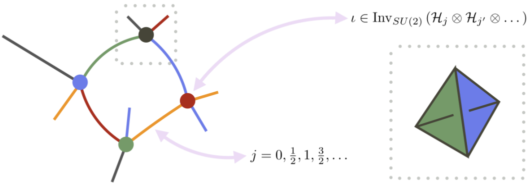
$$\left.\begin{aligned} H_a &= F^i{}_{ab} E_i{}^b = 0 \\ H &= \tfrac{1}{2}\epsilon_i{}^{lm} F^i{}_{ab} E_l{}^a E_m{}^b = 0 \end{aligned}\right\} \quad \textit{generators of hypersurface deformations}$$

- Observables commute with the constraints (gauge generators) $\rightsquigarrow$ no local observables in GR.
- Dirac program: States Ψ are wave-functionals $\Psi[q]$ of the configuration variable q. Particularly neat such functional is a Wilson loop,

$$\Psi_{\alpha,j}[A] = \mathrm{Tr}_j \left[\mathrm{Pexp}\Big(- \int_\alpha A \Big) \right].$$

- In loop gravity, the entire state space is constructed by successively exciting such gravitational Wilson loops out of a vacuum that represents no space at all.

Spin network basis



- Eigenstates of three-geometry are labelled by graphs Γ (combinatorial structure) with spins $\vec{\jmath}$ and intertwiners (Clebsch – Gordan coefficients) $\vec{\iota}$.

$$\Psi = \sum_{\Gamma,\vec{\jmath},\vec{\iota}} \Psi_{\Gamma,\vec{\jmath},\vec{\iota}} \big|\Gamma, \vec{\jmath}, \vec{\iota}\big\rangle$$

- We do not measure microscopic spins and intertwiners, rather components of the Weyl tensor at infinity, mass, energy, angular momentum etc.
- We thus need a description to translate microscopic spins and intertwiners (defined locally) to physical observables (defined non-locally).

$$\big|\Gamma, \vec{\jmath}, \vec{\iota}\big\rangle \stackrel{?}{\longleftrightarrow} \big|M, J, \dots\big\rangle$$

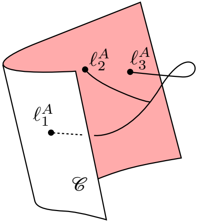
- Two strategies: *(i. relationalism)* Anchor fields at other fields — e.g. using four matter fields φ^μ as material reference systems $x^\mu(\varphi)$. *(ii. quasi-local approach)* Treat the gravitational field in a finite region as a Hamiltonian system. Anchor the observables at a finite boundary, take the boundary to infinity.

Coupling spin-networks to boundaries

- Loop gravity boundary charges: Quantum three-geometry described by spin networks. If they hit a boundary, a surface charge is excited (namely a spinor).

$$\Psi_{\alpha,j}[A] = \mathrm{Tr}_j\Big[\mathrm{Pexp}\Big(-\int_\alpha A\Big)\Big].$$

What is the classical Hamiltonian description for these loop gravity boundary spinors? What is their role in classical GR?

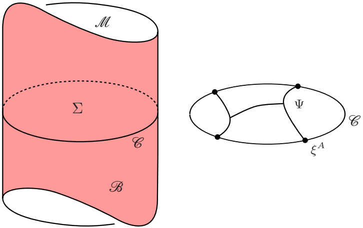


- Suggestive idea: The loop quantum gravity boundary spinors encode gravitational edge modes on the boundary of space time.
  - *Emerged out of spinor representation of LQG [L. Freidel, S. Speziale, E. Livine, Girelli, ww, E. Bianchi et al.]*
  - *Quasi-local realisation of flat space holography [Grumiller, Barnich, Compere,...]*

Outline of the talk

1. Three dimensions
   - *Conformal boundary spinors for quantum gravity in three dimensions*
   - *Quantisation of length without spin networks*
2. Four dimensions
   - *Spinors as gravitational edge modes on null surface boundaries*
   - *Quantisation of area without spin networks*
3. Conclusion

Conformal boundary CFT and 3d euclidean loop quantum gravity



- Setup: Euclidean gravity in three dimensions with vanishing cosmological constant.
- Quasi-local approach: Gravity as a Hamiltonian system in regions with boundaries at finite distance.
- Bulk configuration variables: $SU(2)$ spin connection $A^i{}_a$ and possibly degenerate triad $e^i{}_a$. Corresponding metric: $g_{ab} = \delta_{ij} e^i{}_a e^j{}_b$.

The action in the bulk is topological. EOM given by flatness constraint $F^i = \mathrm{d} \wedge A^i + \frac{1}{2}\epsilon^i{}_{lm} A^l \wedge A^m = 0$ and torsionless condition $\nabla \wedge e^i = 0$.

$$S_{\mathscr{M}}[e, A] = \frac{1}{8\pi G} \int_{\mathscr{M}} e_i \wedge F^i[A].$$

Boundary conditions: Different boundary conditions require then different boundary terms, which, in turn, lead to different boundary field theories.

Goal: Realise quantisation of geometry in terms of a (dual) conformal boundary field theory (for first-order spin connection variables).

- The boundary $\mathscr{B} = \partial\mathscr{M}$ is two-dimensional. In two dimensions, the boundary metric $h_{ab} = \varphi^*_{\mathscr{B}} g_{ab}$ can be always diagonalised by applying appropriate boundary diffeomorphisms.
- The boundary metric is then fully characterised by a conformal factor Ω and a fiducial two-dimensional metric q_{ab}.

Idea: Treat the conformal factor as a dynamical field (from the perspective of the boundary CFT), but fix its conjugate momentum (the trace of the extrinsic curavture) through appropriate boundary conditions. Simplest possibility: $K^a{}_a = 0$.

*E. Witten, A Note On Boundary Conditions In Euclidean Gravity, arXiv:1805.11559v1 (2018).

Conformal boundary conditions

Idea: Treat the conformal factor as a dynamical composite field (from the perspective of the boundary CFT), but fix its conjugate momentum (the trace of the extrinsic curavture). Simplest possibility: $K^a{}_a = 0$.

Conformal boundary conditions

$$\varphi^*_{\mathscr{B}} g_{ab} \in [q_{ab}] \Leftrightarrow \exists \Omega : \mathscr{B} \to \mathbb{R}_+ : \varphi^*_{\mathscr{B}} g_{ab} = \Omega^{-2} q_{ab},$$
$$K = \nabla_a n^a = 0.$$

Nota bene: $K = 0$ is the same as to say that the boundary is a minimal surface (such as a soap film).

*W. Wieland, Conformal boundary conditions, loop quantum gravity and the continuum, JHEP **10** arXiv:1804.08643 (2018).

Key observation: *At the (cylindrical) boundary $\mathscr{B}$ of $\mathscr{M}$ there always exists a spinor ξ^A and a complex-valued one-form $m_a \in \Omega^1(\mathscr{B} : \mathbb{C})$ such that the pull-back of the triad assumes the following form:*

$$\varphi^*_{\mathscr{B}} e^i{}_a = \frac{4\pi G}{\sqrt{2}} \sigma_{AB}{}^i \xi^A \xi^B m_a + \text{cc}.$$

Geometric interpretation

- The dyade $(m_a, \bar{m}_a)$ determines the fiducial boundary metric: $q_{ab} = 2m_{(a}\bar{m}_{b)}$ (boundary indices raised and lowered with q_{ab}, q^{ab}).
- The spinor ξ^A determines the (internal) normal $\vec{n} = \langle\xi|\vec{\sigma}|\xi\rangle / \|\xi\|^2$ to the boundary.
- The norm $\|\xi\|^2 = \delta_{AA'}\xi^A\bar{\xi}^{A'} \equiv \langle\xi|\xi\rangle$ determines the conformal factor.

Conformal factor

$$\varphi^*_{\mathscr{B}} g_{ab} = \Omega^{-2} q_{ab} = (4\pi G)^2 \|\xi\|^4 q_{ab}.$$

Conformal boundary conditions in terms of boundary spinors

We can now neatly express the boundary conditions in terms of the $SU(2)$ boundary spinors.

metric formulation	connection formulation
$\varphi^*_{\mathscr{B}} g_{ab} = \Omega^{-2} q_{ab}$	$\varphi^*_{\mathscr{B}} e^i{}_a = \frac{4\pi G}{\sqrt{2}} \sigma_{AB}{}^i \xi^A \xi^B m_a + \mathrm{cc.}$
$K^a{}_a = 0$	$m^a \mathscr{D}_a \xi^A = 0$

Where we introduced the $SU(2) \times U(1)$ boundary covariant derivative:

- $SU(2) \times U(1)$ boundary covariant derivative: $\mathscr{D}_a \xi^A = D_a \xi^A + \frac{1}{2\mathrm{i}} \Gamma_a \xi^A$.
- $SU(2)$ gauge covariant boundary derivative: $D_a = \varphi^*_{\mathscr{B}} \nabla_a$
- $U(1)$ fiducial boundary spin connection Γ: $\mathrm{d} \wedge m + \mathrm{i}\Gamma \wedge m = 0$.

Boundary conditions derived from bulk plus boundary action

Bulk plus boundary action:

$$\boxed{S[A,e|\xi] = \frac{1}{8\pi G}\int_{\mathscr{M}} e_i \wedge F^i[A] - \frac{\mathrm{i}}{\sqrt{2}}\int_{\mathscr{B}} \Big[\xi_A m \wedge D\xi^A - \mathrm{cc.}\Big]}$$

Variation of the action yields equations of motion in the bulk ($F^i = 0$ and $T^i = \nabla \wedge e^i = 0$) plus boundary conditions:.

The glueing conditions linking the bulk and boundary theories.

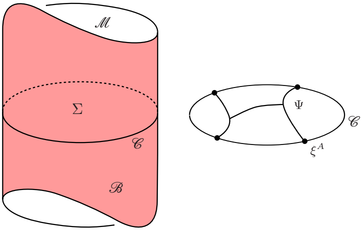
$$\varphi^*_{\mathscr{B}} e^i{}_a = \frac{4\pi G}{\sqrt{2}} \xi^A \xi^B \sigma_{AB}{}^i m_a + \mathrm{cc.}$$

Taking into account the variation of the spinors themselves, we obtain the boundary field equations, namely

$$m \wedge D\xi^A - \frac{1}{2}\mathrm{d}m\,\xi^A = 0 \Leftrightarrow m^a \mathscr{D}_a \xi^A = 0 \Leftrightarrow K^a{}_a = 0.$$

The holomorphicity of the boundary spinor implies that the boundary is a minimal surface. Boundary conditions $=$ boundary EOMs.

Introduce a foliation and evaluate the first variation of the action.



- Pre-symplectic potential:

$$\Theta_\Sigma = -\frac{1}{8\pi G}\int_\Sigma e_i \wedge \mathbb{d}A^i + \\ - \frac{\mathrm{i}}{\sqrt{2}}\int_{\mathscr{C}} (\xi_A m \mathbb{d}\xi^A - \mathrm{cc.}).$$

- Gauge condition: ${A^i}_a = 0$, $m_a = \partial_a z/\sqrt{2}$ is admissible in the cylinder.

This is only partial gauge fixing: residual gauge transformations: $\partial_a \Lambda^i = 0$.
Mode expansion $\xi^A(z) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \xi^A_n z^n$ and symplectic potential:

$$\Theta = \frac{1}{2}\sum_{n=-\infty}^{\infty} \epsilon_{AB}\xi^A_n \mathrm{d}\xi^B_{-n-1} + \mathrm{cc.}$$

Poisson brackets

$$\{\xi^A_n, \xi^B_m\} = \epsilon^{AB}\delta_{m,-n-1}, \qquad \{\bar{\xi}^{A'}_n, \bar{\xi}^{B'}_m\} = \bar{\epsilon}^{A'B'}\delta_{m,-n-1}.$$

To complete the Hamiltonian analysis, we consider gauge transformations and observables.

Simplest: Internal $SU(2)$ transformations, which act in the obvious way,

$$\delta_\Lambda e^i{}_a = \epsilon^i{}_{lm}\Lambda^l e^m{}_a, \quad \delta_\Lambda A^i{}_a = -\nabla_a\Lambda^i, \quad \delta_\Lambda \xi^A = \frac{1}{2\mathrm{i}}\sigma^A{}_{Bi}\Lambda^i\xi^B.$$

The vector fields δ_Λ define degenerate (gauge directions) of the pre-symplectic two-form $\Omega_\Sigma = \mathbb{d}\Theta_\Sigma$ (even for large gauge transformations not vanishing at the boundary),

$$\Omega_\Sigma(\delta_\Lambda, \delta) = 0.$$

The boundary action defines a CFT with vanishing central charge.

The conformal symmetries are generated by vector fields $t^a \in T\mathscr{M}$, whose restrictions to the boundary are conformal Killing vectors:

$$t^a\big|_{\mathscr{B}} \in T\mathscr{B} \,:\, \mathscr{D}_{(a}t_{b)} - \frac{1}{2}q_{ab}\mathscr{D}_c t^c = 0.$$

In the bulk, the diffeomorphisms act through the gauged Lie deriavtive

$$\delta_t e^i = \mathscr{L}_t e^i = t\lrcorner(\nabla \wedge e^i) + \nabla \wedge (t\lrcorner e^i),$$
$$\delta_t A^i = \mathscr{L}_t A^i = t\lrcorner F^i.$$

The boundary fields transfrom with conformal weight $(\frac{1}{2}, 0)$,

$$\delta_t \xi^A := t^a \mathscr{D}_a \xi^A + \frac{1}{2}(\bar{m}^b m^c \mathscr{D}_b t_c)\xi^A.$$

For any such vector field the field variation δ_t is integrable,

$$\Omega_\Sigma(\delta_t, \delta) = -\delta E_t[\mathscr{C}].$$

Quasi-local charge on the boundary $\mathscr{C} = \partial\Sigma$,

$$E_t[\mathscr{C}] = -\frac{\mathrm{i}}{\sqrt{2}} \int_{\mathscr{C}} \Big[t^a m_a \xi_A \mathscr{D} \xi^A - \mathrm{cc.} \Big] = \int_{\mathscr{C}} dv^a t^b T_{ab}.$$

With the conserved and traceless (Brown – York) energy-momentum tensor,

$$T_{ab} = \frac{1}{\sqrt{2}} \Big[m_a m_b \xi_A \bar{m}^c \mathscr{D}_c \xi^A + \mathrm{cc.} \Big] = -\frac{1}{8\pi G} K_{ab}.$$

Consider gauge choice where ${A^i}_a = 0$ and $m_a = \partial_a z/\sqrt{2}$,

$$E_{t_n}[\mathscr{C}] = t_n L_n + \mathrm{cc.}, \quad \text{for:} \quad t^a_n = t_n z^{n+1}\partial^a_z + \bar{t}_n \bar{z}^{n+1}\partial^a_{\bar{z}}.$$

Virasoro generators

$$L_n = \frac{1}{4}\sum_{m=-\infty}^{\infty}(2m+n+1)\,\epsilon_{AB}\xi^A_{-m-n-1}\xi^B_m,$$

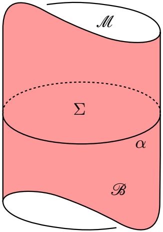
that satisfy the Virasoro algebra with vanishing central charge,

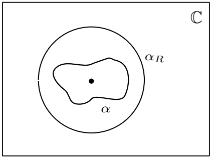
$$\{L_m, L_n\} = (m-n)L_{m+n}.$$

Quantisation of length in three-dimensional euclidean quantum gravity

Super-metric of a loop

We now want to demonstrate length quantisation starting from the field theory in the continuum.





- Consider a loop α winding once around the cylinder $\mathscr{B} = \partial\mathscr{M}$.
- Its physical length $L[\alpha]$ is determined by the conformal factor, proportional to $\|\xi\|^2$.

$$\boldsymbol{L}[\alpha] = \oint_\alpha \mathrm{d}\tau \sqrt{q_{ab}\dot{\gamma}^a\dot{\gamma}^b} \times \Omega$$

We use the mode expansion and find

$$\boldsymbol{L}[\alpha] = 4\pi G \sum_{n,m=-\infty}^{\infty} G^{mn}_{AA'}[\alpha]\xi^A_m\bar{\xi}^{A'}_n .$$

Where we introduced the super-metric on the covariant phase space

$$G^{mn}_{AA'}[\alpha] = \frac{1}{2\pi}\oint_\alpha \mathrm{d}s \left|\frac{\mathrm{d}z}{\mathrm{d}s}\right| z^n\bar{z}^m\delta_{AA'} .$$

Riemann mapping theorem implies that it suffices to show length quantisation for circles in the fiducial background metric $q_{ab} = 2m_{(a}\bar{m}_{b)}$.

For a circle $\alpha_R : |z|^2 = R^2$, the metric is diagonal,

$$G^{mn}_{AA'} = R^{2n+1}\delta_{AA'}\delta^{mn}.$$

Suggesting to introduce the harmonic oscillators for $n \geq 0$,

$$a^A_n = \frac{1}{\sqrt{2}}\Big[R^{n+\frac{1}{2}}\xi^A_n - \frac{\mathrm{i}}{R^{n+\frac{1}{2}}}\delta^A{}_{A'}\bar{\xi}^{A'}_{-n-1}\Big],$$
$$b^A_n = \frac{1}{\sqrt{2}}\Big[\frac{1}{R^{n+\frac{1}{2}}}\xi^A_{-n-1} - \mathrm{i}R^{n+\frac{1}{2}}\delta^A{}_{A'}\bar{\xi}^{A'}_n\Big].$$

Changing R amounts to change the frequency of the harmonic oscillators.

Only non-vanishing Poisson brackets

$$\{a_n^A, \bar{a}_m^{A'}\} = \mathrm{i}\delta_{mn}\delta^{AA'} = \{b_n^A, \bar{b}_m^{A'}\}.$$

Loop Gravity is based on the Ashtekar – Lewandowski vacuum, a state with totally degenerate spatial geometry.

The boundary field theory analogue of this state in the continuum is now simply the Fock vacuum of the oscillators,

$$\forall n \geq 0 : a_n^A |0, \alpha_R\rangle = b_n^A |0, \alpha_R\rangle = 0$$

Choosing a normal ordering, the total length of a loop α_R turns into the sum of two number operators.

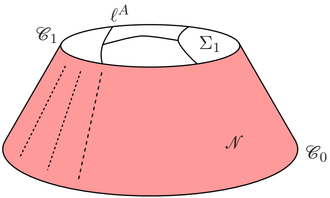
$$\boldsymbol{L}[\alpha_R] = 4\pi G \sum_{n=0}^{\infty} \delta_{AA'} \left[\bar{a}_n^{A'} a_n^A + \bar{b}_n^{A'} b_n^A\right].$$

In three spacetime dimensions, Newton's constant G has dimensions of length. Possible eigenvalues for the circumference of the circle given by

$$0, 4\pi G, 8\pi G, 16\pi G, \dots$$

Four dimensions: Spinors as gravitational edge modes on a null surface

Subsystems of the gravitational field with inner null boundaries $\mathscr{N}$ (all fields assumed to be regular on $\mathscr{N}$, excluding e.g. focal points).



- Boundary consists of partial Cauchy surfaces Σ_0, Σ_1
- and a null surface $\mathscr{N}$ (e.g. isolated Horizon, but this is not necessary).

- The gravitational action consists of bulk plus boundary contributions.
- What counter term shall we put at $\mathscr{N}$? Difficulty: there is now an additional constraint to be imposed—that the boundary is null.
- Working with self-dual Ashtekar variables in the bulk, we will find such a boundary term in terms of boundary spinors coupled to the spin connection in the bulk.

*R. Wald and A. Zoupas, A General Definition of "Conserved Quantities" in General Relativity and Other Theories of Gravity, Phys.Rev. D 61 (2000), arXiv::gr-qc/9911095.
*T. Andrade and D. Marolf, Asymptotic symmetries from finite boxes, Class. Quant. Gravity. 33 (2016), arXiv:1508.02515.

Key observation: Plebański two-form simplifies on null boundaries

On a null surface it is useful to work with forms rather than vectors.
Given a tetrad e^α, we have a hierarchy of p-forms: $e^{\alpha_1} \wedge \cdots \wedge e^{\alpha_p}$.

- Plebański's directed area two-form $\Sigma^{\alpha\beta} = e^\alpha \wedge e^\beta$ splits into self-dual and anti-selfdual components:

$$\begin{pmatrix} \Sigma^A{}_B & \emptyset \\ \emptyset & -\bar{\Sigma}_{A'}{}^{B'} \end{pmatrix} = -\frac{1}{8}[\gamma_\alpha, \gamma_\beta] e^\alpha \wedge e^\beta.$$

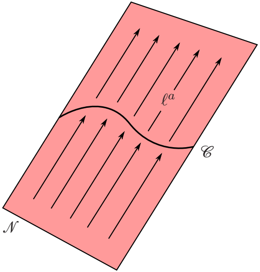
- *On a null surface $\mathcal{N}$, there always exists a spinor $\ell^A : \mathcal{N} \to \mathbb{C}^2$ and a spinor-valued two-form $\eta^A{}_{ab} \in \Omega^2(\mathcal{N} : \mathbb{C}^2)$ such that the pull-back of Σ_{ABab} to the null surface can be parametrised as follows,*

$$\varphi^*_{\mathcal{N}} \Sigma_{ABab} = \ell_{(A} \eta_{B)ab}.$$

*[R. Capovilla, T. Jacobson, J. Dell, L. Mason, J. Plebański , K. Krasnov, H. Urbantke,...]

Intrinsic null geometry in terms of η_{Aab} and ℓ^A

The p-form spinors (η_{Aab}, ℓ^A) determine entire intrinsic geometry of $\mathscr{N}$.



- The spin $(\frac{1}{2}, \frac{1}{2})$ vectorial component $\ell^a \sim \mathrm{i}\ell^A\bar{\ell}^{A'}$ defines the null generators.
- The spin $(1,0)$ tensorial component $\eta_{(A}\ell_{B)}$ defines the pull-back of the self-dual two-form $\varphi^*\Sigma_{AB}$ to $\mathscr{N}$.

- The Lorentz invariant spin $(0,0)$ scalar $\boldsymbol{\varepsilon} = -\mathrm{i}\eta_A\ell^A$ defines the *oriented* area flux of any two-dimensional cross section $\mathscr{C}$ of $\mathscr{N}$

$$\mathrm{Area}_{\boldsymbol{\varepsilon}}[\mathscr{C}] = \int_{\mathscr{C}} \boldsymbol{\varepsilon} = -\mathrm{i} \int_{\mathscr{C}} \eta_A \ell^A.$$

- The pair (η_{Aab}, ℓ^A) determines the intrinsic signature $(0{+}{+})$ metric $q_{ab} = 2m_{(a}\bar{m}_{b)}$ on $\mathscr{N}$ completely.

Two notions of area: densities vs. two-forms

- Metrical area of a cross-section $\mathscr{C}$

$$\mathrm{Area}_g[\mathscr{C}] = \int_{\mathscr{C}} \mathrm{d}s\,\mathrm{d}t \sqrt{\det\begin{pmatrix} g(\partial_s,\partial_s) & g(\partial_s,\partial_t) \\ g(\partial_t,\partial_s) & g(\partial_t,\partial_t). \end{pmatrix}} \geq 0.$$

- Oriented area flux of a cross-section $\mathscr{C}$

$$\mathrm{Area}_{\boldsymbol{\varepsilon}}[\mathscr{C}] = -\mathrm{i}\int_{\mathscr{C}} \eta_A \ell^A \in \mathbb{R}.$$

- Relative sign distinguishes ingoing from outgoing null boundaries.
- Analogous to the two natural volume elements on the manifold,

$$d^4x\sqrt{-\det g_{\mu\nu}} = \pm\frac{1}{4!}\epsilon_{\alpha\beta\mu\nu} e^\alpha \wedge e^\beta \wedge e^\mu \wedge e^\nu.$$

- Boundary spinors (η_{Aab}, ℓ^A) determine the *intrinsic geometry* of $\mathscr{N}$.
- Extrinsic geometry characterised by a $U(1)_{\mathbb{C}}$ boundary connection ω_a and a spinor-valued one-form $\psi^A{}_a$ modulo the equivalence relation

$$\omega_a \sim \omega_a + f\bar{m}_a, \qquad \psi^A{}_a \sim \psi^A{}_a - f\ell^A\bar{m}_a.$$

- Equivalence class $[\omega_a, \psi^A{}_a]$ determines the exterior covariant derivatives (shear+expansion+surface gravity)

$$D\ell^A = +\omega\ell^A + \psi^A,$$
$$D\eta_A = -\omega \wedge \eta_A.$$

- Complexified $U(1)_{\mathbb{C}}$ transformations

$$\eta_{Aab} \longrightarrow \mathrm{e}^{-\zeta}\eta_{Aab}, \quad \psi^A{}_a \longrightarrow \mathrm{e}^{+\zeta}\psi^A{}_a,$$
$$\ell^A \longrightarrow \mathrm{e}^{+\zeta}\ell^A, \qquad \omega_a \longrightarrow \omega_a + \partial_a\zeta.$$

The boundary spinors enter the action through boundary terms.

- Tetradic Hilbert – Palatini action in the bulk,

$$S_{\mathscr{M}}[A,e] = \frac{\mathrm{i}}{8\pi G}\int_{\mathscr{M}} \Sigma_{AB}[e]\wedge F^{AB}[A] + \mathrm{cc}.$$

- $SL(2,\mathbb{C})$-invariant boundary action,

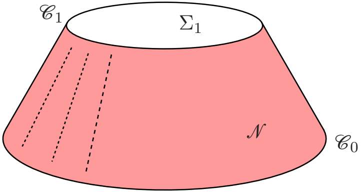
$$S_{\mathscr{N}}[A|\eta,\ell] = \frac{\mathrm{i}}{8\pi G}\int_{\mathscr{N}} \underbrace{\eta_A\wedge D\ell^A}_{\text{“}p\mathrm{d}q\text{”}} + \mathrm{cc}.$$

- bulk plus boundary action

$$S[A,e|\eta,\ell] = S_{\mathscr{M}}[A,e] + S_{\mathscr{N}}[A|\eta,\ell].$$

- The variation of the action determines both the equations of motion and the symplectic potential.

$$\boxed{\delta S = \mathrm{EOM}\cdot\delta + \Theta_{\partial\mathscr{M}}(\delta).}$$



- $\Theta_{\partial\mathscr{M}} = \Theta_{\Sigma_1} + \Theta_{\Sigma_0} + \Theta_{\mathscr{N}}.$
- Covariant Hamiltonian formalism

pre-symplectic two-form: $\Omega_\Sigma = \mathbb{d}\Theta_\Sigma,$

gauge symmetries: $\Omega_\Sigma(\delta,\cdot) = 0,$

Hamilton equations: $\Omega_\Sigma(\delta_H,\delta) = -\delta H.$

*R. Wald and A. Zoupas, A General Definition of "Conserved Quantities" in General Relativity and Other Theories of Gravity, Phys.Rev. D 61 (2000), `arXiv::gr-qc/9911095`.

Covariant pre-symplectic potential for the partial Cauchy surfaces Σ

$$\Theta_\Sigma = \left[-\frac{\mathrm{i}}{8\pi G}\int_{\mathscr{C}} \eta_A \mathbb{d}\ell^A + \frac{\mathrm{i}}{8\pi G}\int_\Sigma \Sigma_{AB} \wedge \mathbb{d}A^{AB}\right] + \mathrm{cc.}$$

Poisson brackets at the two-dimensional corner

$$\big\{\eta_{Aab}(z), \ell^B(z')\big\}_{\mathscr{C}} = 8\pi\mathrm{i}G\delta^B_A \underset{\sim}{\epsilon}_{ab}\delta^{(2)}(z, z').$$

Pre-symplectic structure along the portion of the null surface

$$\Theta_{\mathscr{N}} = -\frac{\mathrm{i}}{8\pi G}\Big[\int_{\mathscr{N}} \underbrace{\eta_A \ell^A \wedge \mathbb{d}\omega}_{\text{`Coulombic part'}} + \underbrace{\eta_A \wedge \mathbb{d}\psi^A}_{\text{`radiative part'}}\Big] + \mathrm{cc.} = \text{“intr. } \wedge \text{ extr. geometry”},$$

with $D\ell^A = \omega\ell^A + \psi^A$ on $\mathscr{N}$.

Fock quantization of area

Inclusion of the Barbero – Immirzi parameter γ

The Immirzi parameter $\gamma > 0$ is a coupling constant in front of the term $e_\alpha \wedge e_\beta \wedge F^{\alpha\beta}[A]$, which can be added to the action without changing the equations of motion. For $\gamma \neq 0$, we must modify then the boundary action as well.

- Bulk action

$$S_{\mathscr{M}}[A,e] = \frac{\mathrm{i}}{8\pi G}\Big[\int_{\mathscr{M}} \frac{\gamma + \mathrm{i}}{\gamma}\,\Sigma_{AB} \wedge F^{AB}\Big] + \mathrm{cc}.$$

- Boundary action for the null surface

$$S_{\mathscr{N}}[A|\eta,\ell] = \frac{\mathrm{i}}{8\pi G}\Big[\int_{\mathscr{N}} \frac{\gamma + \mathrm{i}}{\gamma}\,\eta_A \wedge D\ell^A\Big] + \mathrm{cc}.$$

- Canonical momentum (spinor-valued two-form on the boundary)

$$\boldsymbol{\pi}_A := \frac{\mathrm{i}}{8\pi G}\frac{\gamma + \mathrm{i}}{\gamma}\eta_A.$$

- The Poisson brackets for the boundary variables are

$$\big\{\boldsymbol{\pi}_A(z), \ell_B(z')\big\}_{\mathscr{C}} = \epsilon_{AB}\delta^{(2)}(z,z').$$

- Generator of complexified $U(1)_{\mathbb{C}}$ transformations

$$L = -\frac{1}{2\mathrm{i}}\boldsymbol{\pi}_A\ell^A + \mathrm{cc.} \quad \textit{(generator of U(1) transformations)},$$
$$K = -\frac{1}{2}\boldsymbol{\pi}_A\ell^A + \mathrm{cc.} \quad \textit{(dilatations of the null normal).}$$

- Upon introducing γ, the cross-sectional area is neither L nor K, but

$$\boldsymbol{\varepsilon} = -8\pi G\frac{\gamma}{\gamma+\mathrm{i}}\boldsymbol{\pi}_A\ell^A.$$

- For the area to be real-valued (charge neutral), we have to satisfy the reality conditions,

$$\boldsymbol{\varepsilon} = \bar{\boldsymbol{\varepsilon}} \Leftrightarrow \boxed{K - \gamma L = 0.}$$

- Poisson brackets in the continuum

$$\{\boldsymbol{\pi}_A(z), \ell^B(z')\} = \delta^B_A \delta^{(2)}(z, z').$$

- Strategy: Find creation and annihilation operators and quantise them in the continuum.
- This requires *two* additional structures:

Fiducial hermitian metric: $\delta_{AA'} = \sigma_{AA'\alpha} n^\alpha$,

Fiducial area element: $d^2\Omega = \Omega^2(\vartheta, \varphi) \sin^2\vartheta \, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi$.

- Gravitational Landau operators (half densities)

$$a^A = \frac{1}{\sqrt{2}}\Big[\sqrt{d^2\Omega}\, \delta^{AA'} \bar{\ell}_{A'} - \frac{\mathrm{i}}{\sqrt{d^2\Omega}} \boldsymbol{\pi}^A\Big],$$

$$b^A = \frac{1}{\sqrt{2}}\Big[\sqrt{d^2\Omega}\, \ell^A + \frac{\mathrm{i}}{\sqrt{d^2\Omega}} \delta^{AA'} \bar{\boldsymbol{\pi}}_{A'}\Big].$$

- Poisson brackets

$$\{a^A(z), a^*_B(z')\} = \{b^A(z), b^*_B(z')\} = \mathrm{i}\delta^B_A \delta^{(2)}(z, z').$$

Fock quantisation of the area flux

- Fock vacuum in the continuum

$$\forall z \in \mathscr{C} : a^A(z)\big|\{d^2\Omega, n_\alpha\}, 0\big\rangle = 0,$$
$$b^A(z)\big|\{d^2\Omega, n_\alpha\}, 0\big\rangle = 0.$$

- Imposition of the reality conditions:

$$\hat{L}(z) = \frac{1}{2}\big[a^\dagger_A(z)a^A(z) - b^\dagger_A(z)b^A(z)\big],$$
$$\hat{K}(z) = \frac{1}{2\mathrm{i}}\big[a_A(z)b^A(z) - \mathrm{hc.}\big],$$
$$\boxed{\big[\hat{K}(z) - \gamma\hat{L}(z)\big]\Psi_{\mathrm{phys}} = 0.}$$

- $\hat{K}$ is a squeeze operator, $\hat{L}$ plays the role of intrinsic spin.
- Physical states exhibit quantization of area

$$\widehat{\mathrm{Area}_{\boldsymbol{\varepsilon}}[\mathscr{C}]}\Psi_{\mathrm{phys}} = 4\pi\gamma G\int_{\mathscr{C}}\big[a^\dagger_A a^A - b^\dagger_A b^A\big]\Psi_{\mathrm{phys}}.$$

- Possible measurement outcomes for cross-sectional area of $\mathscr{N}$

$$a_j = \frac{8\pi\gamma\,\hbar G}{c^3}j, \quad 2j \in \mathbb{Z}.$$

Conclusion and Outlook

Summary

- We started with a heuristic argument: In LQG, the quantum states of geometry are built from gravitational Wilson lines for the spin connection. If these Wilson lines hit a boundary, they excite a surface charge, namely a spinor sitting at a puncture.
- We then found the classical interpretation for these surface spinors: The LQG boundary spinors appear already at the classical level as gravitational edge modes in the Hamiltonian formalism *in domains bounded by null surfaces*.
- Quantisation of area in conventional Fock space: The generator of dilatations of the null normal is simply the cross-sectional area. We then quantised the area by quantising the boundary spinors using a conventional Fock representation. Upon introducing the Immirzi parameter γ, we reproduced the LQG quantisation of area without ever introducing spin networks or discretizations of space.
- Goals ahead: We now have *two* representation of quantum geometry, (i) discrete spin network representation and (ii) boundary Fock representation. Understand algebra of observables, relation to twistor theory, scattering amplitudes.